

Probability of getting a lucky short-exposure image through turbulence*

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(Received 2 May 1977)

In short-exposure imaging through turbulence, there is some probability that the image will be nearly diffraction limited because the instantaneous wave-front distortion over the aperture was negligible. A number of years ago in a rather brief paper, Hufnagel (1966) argued heuristically that the probability of getting a good image would decrease exponentially with aperture area. This paper undertakes a rigorous quantitative analysis of the probability. We find that the probability of obtaining a good short-exposure image is $\text{Prob} \approx 5.6 \exp[-0.1557 (D/r_0)^2]$ (for $D/r_0 \geq 3.5$), where D is the aperture diameter and r_0 is the coherence length of the distorted wave front, as defined by Fried (1967). A good image is taken to be one for which the squared wave-front distortion over the aperture is 1 rad^2 or less. The analysis is based on the decomposition of the distorted wave front over the aperture, in an orthonormal series with randomly independent coefficients. The orthonormal functions used are the eigenfunctions of a Karhunen-Loève integral equation. The integral equation is solved using a separation of variables into radial and azimuthal dependence. The azimuthal dependence was solved analytically and the radial, numerically. The first 569 radial eigenfunctions and eigenvalues were obtained. The probability of obtaining a good short-exposure image corresponds to a hyperspace integral in which the spatial dimensions are the independent random coefficients in the orthonormal series expansion. It is equal to the probability that a randomly chosen point in the hyperspace will lie within a hypersphere of unit radius, the points in the hyperspace being randomly chosen in accordance with the product of independent Gaussian probability distribution—one distribution for each dimension. The variance of these distributions is directly proportional to the eigenvalues of the Karhunen-Loève equation. This hyperspace integral (involving up to several hundred dimensions) has been evaluated using Monte Carlo techniques.

I. INTRODUCTION

Atmospheric turbulence distorts a wave passing through it so that an image of an object seen through such turbulence may be degraded significantly below the diffraction limit of the imaging system's aperture. The length r_0 which is determined by the strength of turbulence over the propagation path, the path length, and the wavelength sets the achievable resolution. Whereas the diffraction-limited resolution might be considered to be λ/D , the turbulence-limited resolution is equal to λ/r_0 . No matter how large the aperture is, the average resolution achieved on imaging through turbulence will not exceed the λ/r_0 limit. This characterizes what is often referred to as the long-exposure image resolution. [For very short exposures, somewhat better average resolution, $\lambda/(3.4 r_0)$, is possible.]

This turbulence limit is a limit on the average performance. It is significant to note that at each instant of time a randomly distorted wave front is received by the imaging system aperture, and the randomly distorted and spread image is formed. The average resolution λ/r_0 refers to the average effect of this distortion and spreading. At some instant of time, the instantaneous distortion of the wave front may be very severe, i.e., there will be a great deal of "corrugation" of the wave front. At other moments, the wave-front distortion may be relatively slight. There is a finite probability that at some particular instant of time, the wave-front distortion will be almost negligible. A short-exposure image formed at that time will appear to be almost diffraction limited. The question we address in this paper is what is the probability that at a particular instant of

time the wave-front distortion over the imaging system aperture will, in fact, be almost negligible. The inverse of this probability is the average number of short-exposure pictures we have to take to get a good one.

A number of years ago, this question was briefly addressed by Hufnagel.¹ He utilized rather heuristic arguments, considering speckle in the focal plane, the average spread, image size, and the Rayleigh distribution of speckle, and based on these considerations argued that the probability of getting a good image ought to vary as a negative exponential function of the aperture diameter squared. Because the arguments were heuristic, he was not able to provide meaningful constants for this relationship. We shall show in this paper that the probability of getting a good image, i.e., the probability that at some instant the wave-front distortion over the imaging system aperture is essentially negligible, is a negative exponential function of the aperture diameter over r_0 ratio squared, for aperture diameters larger than about $3.5 r_0$. (Here r_0 is the turbulence-limited coherence diameter, as defined by Fried.²) We shall show that the probability of getting a good image is adequately represented by the expression

$$\text{Prob} \approx 5.6 \exp[-0.1557 (D/r_0)^2].$$

In Sec. II, we shall outline our approach to a rigorous analysis of this problem. The sections after that will then present the various phases of the analysis and numerical computations.

II. ANALYTIC APPROACH

The basis of our treatment of the problem of calculating the probability of getting a good short-exposure image

is through a decomposition of the random wave front into an orthonormal series of independent components. By developing the statistics of this series representation, we are then able to calculate the probability that a particular sample of the wave-front distortion will correspond to essentially no significant distortion. This is because of the fact that a sample in which the wave-front distortion is essentially negligible has a series representation in which the coefficients are all very small. It is thus simply a matter of calculating the probability that all of the coefficients of the orthonormal series at some instant of time will be very small.

To carry out the detailed analysis required, we start with a functional representation of a sample of a randomly distorted wave front over the imaging system aperture. Because wave-front tilt, as distinguished from the higher-order forms of wave-front distortion, i.e., from wave-front corrugation, does not degrade the resolution of the image of a point source (in a short-exposure), but merely shifts it, we calculate the average wave-front tilt of this random sample and subtract it from the random wave-front distortion. The difference is the effective wave-front distortion, whose statistics we are concerned with.

Having established the function representation for the effective wave-front distortion, we then wish to decompose this tilt-free randomly distorted wave-front sample over the aperture in terms of some orthonormal series. There are many possible orthonormal series which can be defined for uniform weighting over the aperture. The expression for the decomposition of the tilt-free wave-front distortion is essentially the same no matter which series we use. The basic expression that calculates the coefficients for the series representation of the tilt-free wave front is a weighted integral of the random tilt-free wave-front distortion function. As a consequence of the fact that the tilt-free wave-front distortion is a Gaussian random function, the coefficients of the series are Gaussian random variables, since the integral expression is a linear function of the distortion function.

We select which of the infinity of possible orthonormal series is to be used in this decomposition by imposing the requirement that the random coefficients should be statistically independent of each other. Through appropriate manipulations, we are able to show that this requirement gives rise to a definition of the functions in the orthonormal series in terms of the Karhunen-Loève integral equation. The kernel of the integral equation is expressible as a function of the statistics of wave-front distortion. The Karhunen-Loève integral equation is homogeneous and gives rise to a set of eigenvalues, one for each of the eigenfunction solutions. The eigenfunctions are the functions that make up the orthonormal set. The eigenvalues are proportional to the mean square value of the random coefficients in the orthonormal series decomposition of the tilt-free random wave-front distortion.

The evaluation of the eigenvalues, and thus of the mean square random coefficients, is the key to the evaluation of the probability of getting a good short-exposure image. The random coefficients, as we noted before, are Gauss-

ian random variables. Their mean values are zero, and thus their mean square values, which are calculated from the eigenvalues, are their variances—which completely define the distribution of expected values of these random coefficients.

The significant point will be established that the mean square wave-front distortion averaged over the aperture for a particular sample of the randomly distorted wave-front is directly proportional to the sum of the squares of the random coefficients for the wave-front sample. Thus, the probability of getting a good short-exposure image is reduced to the calculation of the probability that all of the random coefficients will at some instant have a small enough magnitude. We take as our criterion for small enough wave-front distortion the requirement that the tilt-free wave-front distortion squared, averaged over the aperture, will be less than or equal to one radian squared. This leads directly to a constraint on the sum of the squares of the random coefficients. Since we know the probability distribution for all of the random coefficients, it is a straightforward matter to then calculate the probability that their values squared and summed will be appropriately bounded.

In the following work, we rely on analytic techniques to develop the Karhunen-Loève integral equation and the basic expression defining its kernel, and similarly rely on analytic techniques for the formulation of the probability integral we wish to evaluate. The solutions of the Karhunen-Loève integral equation to obtain the eigenvalues (and the eigenfunctions) and the evaluation of the probability integral are of necessity carried out numerically. In Sec. III, we start the analysis treating the formulation leading up to the expression of the Karhunen-Loève equation.

III. DEVELOPMENT OF THE KARHUNEN-LOÈVE EQUATION

We shall let $\phi(\mathbf{r})$ denote the random phase at a point \mathbf{r} associated with the random wave front distortion at the aperture of the imaging system. Here \mathbf{r} is a two-dimensional variable denoting position in the plane of the imaging system aperture. We use the function $W(\mathbf{r}, D)$, where

$$W(\mathbf{r}, D) = \begin{cases} 1, & \text{if } |\mathbf{r}| \leq \frac{1}{2} D, \\ 0, & \text{if } |\mathbf{r}| > \frac{1}{2} D, \end{cases} \quad (1)$$

to define a circular aperture centered at the origin and having a diameter D . Drawing on our previously published results, Fried,³ we can write for the corresponding average phase over the aperture $\bar{\phi}$,

$$\bar{\phi} = (\frac{1}{4} \pi D^2)^{-1} \int d\mathbf{r} W(\mathbf{r}, D) \phi(\mathbf{r}), \quad (2)$$

and for the average tilt over the aperture α ,

$$\alpha = (\frac{1}{8} \pi D^4)^{-1} \int d\mathbf{r} W(\mathbf{r}, D) \mathbf{r} \phi(\mathbf{r}). \quad (3)$$

The effective wave-front distortion, as far as the quality of the image is concerned, is the random wave-front distortion $\phi(\mathbf{r})$, less the average phase and a linear function,

$\alpha \cdot \mathbf{r}$, corresponding to the average tilt. The effective wave-front distortion, $\varphi(\mathbf{r}; D)$ is given by

$$\varphi(\mathbf{r}; D) = \phi(\mathbf{r}) - \bar{\phi} - \alpha \cdot \mathbf{r}. \quad (4)$$

(We note that since $\bar{\phi}$ and α are functions of D , it is appropriate to show φ as having a D dependence.)

The statistics of the random wave-front distortion $\phi(\mathbf{r})$ is adequately described for our purposes in terms of the phase structure function \mathcal{D} , where

$$\mathcal{D}(|\mathbf{r} - \mathbf{r}'|) = \langle |\phi(\mathbf{r}) - \phi(\mathbf{r}')|^2 \rangle. \quad (5)$$

For our calculations, all relevant theory concerning propagation through turbulence is adequately represented by taking note of the $\frac{5}{3}$ -power dependence established by Tatarski⁴ for the phase structure function. This is conveniently represented by the expression given by Fried²

$$\mathcal{D}(|\mathbf{r} - \mathbf{r}'|) = 6.88(|\mathbf{r} - \mathbf{r}'|/\mathbf{r}_0)^{5/3}. \quad (6)$$

The mean square difference of random phase is seen to vary as the $\frac{5}{3}$ power of the separations of the points at which the phase is measured. The quantity \mathbf{r}_0 , which as we noted earlier is determined by propagation conditions, is given by the expression

$$\mathbf{r}_0 = \left(0.423 k^2 \int_{\text{path}} ds C_N^2(s/L)^{5/3} \right)^{-3/5}, \quad (7)$$

where C_N^2 is the refractive-index structure constant (a measure of the optical strength of turbulence along the propagation path), L is the total propagation path length, and s runs from zero at the (point) source to L at the aperture plane, where \mathbf{r}_0 is measured. \mathbf{r}_0 determines the actual "magnitude" of the wave-front distortion.

What we are ultimately going to be concerned with is the covariance function for the effective wave-front distortion $\varphi(\mathbf{r}; D)$. This covariance function is given by the expression

$$C_\varphi(|\mathbf{r} - \mathbf{r}'|; D) = \langle \varphi^*(\mathbf{r}; D) \varphi(\mathbf{r}'; D) \rangle. \quad (8)$$

It can be evaluated in terms of the structure function \mathcal{D} by making use of Eqs. (2), (3), (4), and (5). We shall return later to the evaluation of the covariance function C_φ .

We consider an orthonormal series $\{f_n(\mathbf{r}; D)\}$, where f_n is the n th-term in the orthonormal series. Because the terms are orthonormal with uniform weighting over the aperture area, it follows that

$$\int d\mathbf{r} W(\mathbf{r}, D) f_n(\mathbf{r}; D) f_{n'}(\mathbf{r}; D) = \begin{cases} 1, & \text{if } n = n', \\ 0, & \text{if } n \neq n'. \end{cases} \quad (9)$$

Since the orthonormal series is by definition complete, we can represent the effective wave-front distortion φ in terms of a series based on $\{f_n\}$. Thus we write

$$\varphi(\mathbf{r}; D) = \sum_n \beta_n f_n(\mathbf{r}; D). \quad (10)$$

Inasmuch as the effective wave-front distortion φ is a random function, we expect the coefficients of the series β_n to be random variables. If we multiply both sides of

Eq. (10) by $f_n^*(\mathbf{r}; D)$ and integrate over the aperture, then making use of Eq. (9), we obtain the result

$$\beta_n = \int d\mathbf{r} W(\mathbf{r}; D) f_n^*(\mathbf{r}, D) \varphi(\mathbf{r}; D). \quad (11)$$

This is the standard expression for the coefficients in any type of orthonormal series decomposition.

We note that since the coefficient β_n is a linear function of the effective wave-front distortion, $\varphi(\mathbf{r}; D)$, and since the effective wave-front distortion is a Gaussian random function, it follows that β_n is a Gaussian random variable. [Note: Propagation theory, cf. Tatarski,⁴ tells us that the wave-front distortion ϕ is a Gaussian random function. From Eqs. (2) and (3), it follows that $\bar{\phi}$ and α , which are linear functions of ϕ , are Gaussian random variables. Since the linear combination of Gaussian random functions and Gaussian random variables gives rise to a Gaussian random function, it follows from Eq. (4) that φ is a Gaussian random function.]

At this point, we are ready to introduce the requirement that the random variables β_n be statistically independent. We write this as

$$\langle \beta_n^* \beta_{n'} \rangle = \begin{cases} B_n^2(D), & \text{if } n = n', \\ 0, & \text{if } n \neq n', \end{cases} \quad (12)$$

where $B_n^2(D)$ is the variance associated with the random variable β_n . We show B_n^2 as a function of D , since, as will be seen, the magnitude of the random variable β_n is dependent on the aperture diameter.

Equation (12) provides the basis for developing equations which define the orthonormal series $\{f_n(\mathbf{r}; D)\}$. To do this, we define the quantity \mathcal{G} in accordance with the equation

$$\mathcal{G} = \left\langle \int d\mathbf{r}' W(\mathbf{r}', D) \varphi^*(\mathbf{r}'; D) \varphi(\mathbf{r}; D) f_n(\mathbf{r}'; D) \right\rangle. \quad (13)$$

By the simple procedure of interchanging the order of integration and ensemble averaging, and making use of Eq. (8), we can recast Eq. (13) in the form

$$\mathcal{G} = \int d\mathbf{r}' W(\mathbf{r}', D) C_\varphi(|\mathbf{r} - \mathbf{r}'|; D) f_n(\mathbf{r}'; D). \quad (14)$$

Starting from Eq. (13), and making use of Eq. (10) to allow replacement of $\varphi^*(\mathbf{r}'; D)$, we obtain

$$\mathcal{G} = \left\langle \int d\mathbf{r}' W(\mathbf{r}'; D) \varphi(\mathbf{r}; D) \times \sum_{n'} \beta_{n'}^* f_{n'}^*(\mathbf{r}'; D) f_n(\mathbf{r}'; D) \right\rangle. \quad (15)$$

By the expedient of interchanging the order of summation and integration, and then making use of the orthonormal property expressed in Eq. (9) to allow the integral to be evaluated, we can cast Eq. (15) in the form

$$\mathcal{G} = \left\langle \varphi(\mathbf{r}; D) \sum_{n'} \beta_{n'} \int d\mathbf{r}' W(\mathbf{r}', D) f_{n'}^*(\mathbf{r}'; D) f_n(\mathbf{r}'; D) \right\rangle = \langle \varphi(\mathbf{r}; D) \beta_n^* \rangle. \quad (16)$$

Now if we again make use of Eq. (10) to replace φ by

its series representation, and then interchange the order of summation and ensemble averaging, and finally make use of Eq. (12) to facilitate performance of the n' summation, we obtain

$$\mathcal{E} = \left\langle \sum_{n'} \beta_n f_n(\mathbf{r}; D) \beta_n^* \right\rangle = \sum_{n'} f_n(\mathbf{r}; D) \langle \beta_n^* \beta_n \rangle = B_n^2(D) f_n(\mathbf{r}; D). \quad (17)$$

Equating the right-hand side of Eqs. (14) and (17), we obtain the Karhunen-Loève homogeneous integral equation

$$\int d\mathbf{r}' W(\mathbf{r}', D) C_\phi(|\mathbf{r} - \mathbf{r}'|; D) f_n(\mathbf{r}'; D) = B_n^2(D) f_n(\mathbf{r}; D). \quad (18)$$

Our problem at this point is to solve the integral equation for the eigenfunctions $f_n(\mathbf{r}; D)$, and the corresponding eigenvalues $B_n(D)$.

IV. REDUCTION AND SOLUTION OF THE KARTHUNEN-LOÈVE INTEGRAL EQUATION

The kernel of the Karhunen-Loève integral equation, Eq. (18), can be written in the form

$$\begin{aligned} C_\phi(|\mathbf{r} - \mathbf{r}'|; D) = & \langle [\phi(\mathbf{r}) - \bar{\phi} - \alpha \cdot \mathbf{r}]^* [\phi(\mathbf{r}') - \bar{\phi} - \alpha \cdot \mathbf{r}'] \rangle \\ = & \langle \phi^*(\mathbf{r}) \phi(\mathbf{r}') \rangle - \langle \phi^*(\mathbf{r}) \bar{\phi} \rangle - \langle \phi(\mathbf{r}') \bar{\phi}^* \rangle + \langle \bar{\phi}^* \bar{\phi} \rangle \\ - & \langle \phi^*(\mathbf{r}) \alpha \cdot \mathbf{r}' \rangle - \langle \phi(\mathbf{r}') \bar{\alpha}^* \cdot \mathbf{r} \rangle + \langle \bar{\phi}^* \alpha \cdot \mathbf{r}' \rangle \\ + & \langle \bar{\phi} \alpha \cdot \mathbf{r} \rangle + \langle \bar{\alpha}^* \cdot \mathbf{r} \alpha \cdot \mathbf{r}' \rangle. \end{aligned} \quad (19)$$

If we make the change of variables

$$\mathbf{x} = \mathbf{r}/D, \quad (20a)$$

$$\mathbf{x}' = \mathbf{r}'/D, \quad (20b)$$

we find that the D and r_0 dependence of the kernel can be extracted, and we can write

$$C_\phi(|\mathbf{r} - \mathbf{r}'|; D) = (D/r_0)^{5/3} \mathcal{C}(|\mathbf{x} - \mathbf{x}'|). \quad (21)$$

If we further write

$$f_n(\mathbf{r}; D) \equiv \tilde{f}_n(\mathbf{x}), \quad (22)$$

$$B_n^2(D) = D^2(D/r_0)^{5/3} \mathfrak{B}_n^2, \quad (23)$$

$$W(\mathbf{r}; D) = \mathfrak{W}(\mathbf{x}) = \begin{cases} 1, & \text{if } |\mathbf{x}| \leq \frac{1}{2}, \\ 0, & \text{if } |\mathbf{x}| > \frac{1}{2}, \end{cases} \quad (24)$$

then we can recast Eq. (18) in the form

$$\int d\mathbf{x}' \mathfrak{W}(\mathbf{x}') \mathcal{C}(|\mathbf{x} - \mathbf{x}'|) \tilde{f}_n(\mathbf{x}') = \mathfrak{B}_n^2 \tilde{f}_n(\mathbf{x}). \quad (25)$$

The key to this extraction of the D and r_0 dependence from the Karhunen-Loève integral equation is our ability to extract a $(D/r_0)^{5/3}$ from C_ϕ , as expressed in Eq. (21). After considerable algebraic manipulations, starting from Eq. (19), and making use of Eq. (6), as appropriate, we find that we can write C_ϕ in the form

$$\begin{aligned} C_\phi(|\mathbf{r} - \mathbf{r}'|; D) \\ = & (D/r_0)^{5/3} \{ -\mathfrak{G}_0(|\mathbf{r}/D - \mathbf{r}'/D|) + \mathfrak{G}_1(r/D) \\ + & \mathfrak{G}_1(r'/D) - \mathfrak{G}_2 + (r'/D) \cos \theta' \mathfrak{G}_3(r/D) \end{aligned}$$

$$+ (r/D) \cos \theta' \mathfrak{G}_3(r'/D) - (r'/D) \cos \theta' \mathfrak{G}_4 \}, \quad (26)$$

where

$$\mathfrak{G}_0(x) = 3.44 x^{5/3}, \quad (27a)$$

$$\begin{aligned} \mathfrak{G}_1(x) = & 3.44 \left(\frac{1}{4}\pi\right)^{-1} \int_0^{1/2} dx'' x'' \\ \times & \int_0^{2\pi} d\theta'' (x^2 + x''^2 - 2xx'' \cos \theta'')^{5/6}, \end{aligned} \quad (27b)$$

$$\mathfrak{G}_2(x) = 8 \int_0^{1/2} dx'' x'' \mathfrak{G}_1(x''), \quad (27c)$$

$$\begin{aligned} \mathfrak{G}_3(x) = & 3.44 \left(\frac{1}{64}\pi\right)^{-1} \int_0^{1/2} dx'' x''^2 \int_0^{2\pi} d\theta'' \cos \theta'' \\ \times & (x^2 + x''^2 - 2xx'' \cos \theta'')^{5/6}, \end{aligned} \quad (27d)$$

$$\mathfrak{G}_4(x) = 64 \int_0^{1/2} dx'' x''^2 \mathfrak{G}_3(x''). \quad (27e)$$

This constitutes the basis for writing C_ϕ in the form shown on the right-hand side of Eq. (21).

Gathering all these results together, we would write \mathcal{C} for use in Eq. (25) in the form

$$\begin{aligned} \mathcal{C}(|\mathbf{x} - \mathbf{x}'|) = & \mathfrak{G}_0(|\mathbf{x} - \mathbf{x}'|) + \mathfrak{G}_1(x) + \mathfrak{G}_1(x') - \mathfrak{G}_2 \\ + & x' \cos \theta' \mathfrak{G}_3(x) + x \cos \theta' \mathfrak{G}_3(x') - xx' \cos \theta' \mathfrak{G}_4. \end{aligned} \quad (28)$$

The quantity θ' appearing in Eqs. (26) and (28) represents the angle between the vectors \mathbf{r} and \mathbf{r}' , and also between the vectors \mathbf{x} and \mathbf{x}' .

Equation (25) is now essentially ready for us to start work on developing its solutions. Examination of the kernel and cognizance of the awkward powers involved (i.e., the $\frac{5}{3}$ power and the $\frac{5}{6}$ power) makes it clear that an analytic solution is not likely, and that numerical techniques will be necessary. The fact that the integral equation involves a two-dimensional integral makes the task an exceedingly difficult one for a digital computer. However, it is not strictly necessary to work with the two-dimensional integral equation, and in fact, we can cast our job in terms of obtaining a solution to a one-dimensional integral equation. To do this, we make use of the fact that working in (x, θ) polar coordinates where

$$\mathbf{x} \equiv (x, \theta), \quad (29)$$

we can show that a separation of variables is possible according to which $\tilde{f}_n(\mathbf{x})$ can be expressed as a product of a function of x and a function of θ . Moreover, we can show that the function of θ is an exponential. We write

$$\tilde{f}_n(\mathbf{x}) \equiv \mathfrak{R}_p^q(x) \exp(iq\theta). \quad (30)$$

We shall see that it is a member of an orthonormal set defined by a Karhunen-Loève one-dimensional integral equation, where $\mathfrak{R}_{p,q}(x)$ is a radial function to be determined.

Where previously n was an ordinal number arranging the eigenfunctions and eigenvalues, we now replace it with the ordinal number pair (p, q)

$$n \equiv (p, q), \quad (31)$$

where q is the ordinal number for the azimuthal dependence. There is a different set of radial functions for

each azimuthal dependence—hence the q associated with $\mathcal{R}_q(x)$. p is the ordinal number for the various radial functions in a given set associated with a particular q .

To show that Eq. (30) is valid, it is merely necessary to substitute it into the left-hand side of Eq. (25) and show that the results can be cast in a form which is consistent with the right-hand side of Eq. (25), as interpreted in terms of Eq. (30), i.e., we show consistency in our assumption. [Because the functions $\mathcal{R}_{p,q}(x)$ and $\exp(iq\theta)$ define complete sets, this is a necessary and sufficient proof.] To proceed, we define the modified kernel

$$\mathcal{R}_q(x, x') = x' \int_0^{2\pi} d\theta' \mathcal{G}(x, x', \theta') \exp(iq\theta'), \quad (32)$$

and the alternate notation for the eigenvalue

$$\mathcal{B}_n^2 \equiv \mathcal{B}_{p,q}^2. \quad (33)$$

Now, substituting Eq. (30) into the left-hand side of Eq. (25), we get

$$\begin{aligned} & \int d\mathbf{x}' \mathcal{B}(\mathbf{x}') \mathcal{G}(|\mathbf{x} - \mathbf{x}'|) \mathcal{F}_n(\mathbf{x}') \\ &= \int_0^{1/2} dx' x' \int_0^{2\pi} d\theta' \mathcal{G}(x, x', \theta' - \theta) \mathcal{R}_{p,q}(x') \exp(iq\theta') \\ &= \exp(iq\theta) \int_0^{1/2} dx' \mathcal{R}_q(x, x') \mathcal{R}_p^q(x') \\ &= \mathcal{B}_{p,q}^2 \mathcal{R}_p^q(x) \exp(iq\theta). \end{aligned} \quad (34)$$

We note that in the last form presented in Eq. (34), the result interpreted in terms of Eq. (30) is consistent with the right-hand side of Eq. (25). This proves the validity of the separation of variables expressed by Eq. (30). In developing the final result in Eq. (34), we have made use of the fact that the final dx' integral is simply a function of x .

The function \mathcal{R}_p^q , which is the radial dependence of \mathcal{F}_n as indicated in Eq. (30), is defined by the requirement that it satisfy the one-dimensional homogeneous integral equation

$$\int_0^{1/2} dx' \mathcal{R}_q(x, x') \mathcal{R}_p^q(x') = \mathcal{B}_{p,q}^2 \mathcal{R}_p^q(x). \quad (35)$$

The key point here is, of course, our freedom to choose \mathcal{R}_p^q to satisfy this equation and use the functions so defined in Eq. (30).

We note that the kernel of this one-dimensional homogeneous integral equation can be written in terms of the functions defined in Eqs. (27a)–(27e). If we substitute Eq. (28) into Eq. (32) and perform what simplifications are possible, we obtain

$$\begin{aligned} \mathcal{R}_0(x, x') &= -x' \int_0^{2\pi} d\theta' \mathcal{G}_0([x^2 + x'^2 - 2xx' \cos \theta']^{1/2}) \\ &\quad + 2\pi x' [\mathcal{G}_1(x) + \mathcal{G}_1(x') - \mathcal{G}_2], \end{aligned} \quad (36a)$$

$$\begin{aligned} \mathcal{R}_{\pm 1}(x, x') &= x' \int_0^{2\pi} d\theta' \mathcal{G}_0([x^2 + x'^2 - 2xx' \cos \theta']^{1/2}) \exp(i\theta) \\ &\quad + \pi x' [x' \mathcal{G}_3(x) + \mathcal{G}_3(x') - xx' \mathcal{G}_4], \end{aligned} \quad (36b)$$

$$\mathcal{R}_q(x, x') = x' \int_0^{2\pi} \mathcal{G}_0([x^2 + x'^2 - 2xx' \cos \theta']^{1/2}) \exp(iq\theta), \quad (36c)$$

$$q = \pm 2, \pm 3, \pm 4, \dots$$

Our problem now is the solution of the one-dimensional homogeneous integral equation presented by Eq. (35), with the kernel defined by Eqs. (36a), (36b), or (36c).

For our numerical analysis, we have carried out this solution for the first 42 values of q , i.e., $q = 0$ to $q = 41$, using a 20-point radial space in the interval 0 to $\frac{1}{2}$. Sorting the eigenvalues in accordance with their magnitude, we find that we have obtained at least the first 569 of the eigenvalues. (Though more than 569 eigenvalues were actually calculated, we have no assurance that the $p = 1$, $q = \pm 42$ eigenvalue which we did not obtain does not rank above the 570th of the eigenvalues that we did obtain. We are, however, sure that it is smaller than the 569th eigenvalue which is for $p = 1$, $q = 41$. Hence our termination of the list at that point.) As a check on the accuracy of our calculations and the completeness of our list, we have been able to separately show, making use of earlier work, Fried,³ quite distinct from this Karhunen-Loève decomposition, that the infinite sum of eigenvalues must have the value of

$$\sum_{n=1}^{\infty} \mathcal{B}_n = 0.1056. \quad (37a)$$

We note that

$$\sum_{n=1}^{569} \mathcal{B}_n = 0.1047 = 0.9915 \sum_{n=1}^{\infty} \mathcal{B}_n, \quad (37b)$$

which gives us fairly strong assurance that our list is essentially complete and our computations reasonably accurate. In Table I, we list these eigenvalues and their cumulative sum for the first 100 eigenvalues. We note that because the integral equation and kernel are symmetric between q and $-q$, it is not necessary for us to solve with negative values of q . The eigenvalues obtained for all values of q other than $q = 0$ are doubly degenerate since they correspond to both $+q$ and $-q$. Accordingly, in Table I we have counted twice those eigenvalues for which q is not equal to zero, though they are listed only once, and the cumulative sum as shown corresponds to the double addition of those eigenvalues.

With the eigenvalues thus tabulated, we are ready to turn our attention to the formulation and evaluation of the probabilistic integral determining the probability that at any instant the random wavefront distortion will yield a good short-exposure image.

V. PROBABILITY FORMULATION AND EVALUATION

The square of the effective wave-front distortion, $\varphi(\mathbf{r}; D)$, averaged over the aperture can be written

$$\Delta^2 = (\frac{1}{4} \pi D^2)^{-1} \int d\mathbf{r} W(\mathbf{r}, D) |\varphi(\mathbf{r}; D)|^2. \quad (38)$$

If we use Eq. (10) to replace φ in Eq. (38), and then interchange the order of summation and integration, we get

TABLE I. Eigenvalue listing, $\mathfrak{B}_n^2 = \mathfrak{B}_{p,q}^2$. Eigenvalues are ranked by magnitude, with ordinal number n assigned accordingly to rank. Where the eigenvalue is degenerate (i.e., $q \neq 0$), two values of n are assigned. This is indicated in the (p, q) numbering by listing both a positive and a negative value for q . The p values shown for each eigenvalue represent an ordinal ranking by magnitude for each value of the azimuthal number q . The eigenvalues are listed along with the cumulative sum of the eigenvalues. Whenever the eigenvalue is degenerate, the corresponding cumulative sum represents the addition of the eigenvalue twice.

n	Eigenvalue \mathfrak{B}_n^2	Cumulative sum $\sum_{i=1}^n \mathfrak{B}_i^2$	p, q
1, 2	0.018765000	0.037530000	1 2, -2
3	0.018733000	0.056263000	1 0
4, 5	0.005215500	0.066694000	1 3, -3
6, 7	0.005180000	0.077054000	1 1, -1
8, 9	0.002153400	0.081360800	1 4, -4
10, 11	0.001645600	0.084652000	2 2, -2
12	0.001632200	0.086284200	2 0
13, 14	0.001084800	0.088453800	1 5, -5
15, 16	0.000773650	0.090001100	2 3, -3
17, 18	0.000757140	0.091515380	2 1, -1
19, 20	0.000617730	0.092750840	1 6, -6
21, 22	0.000428150	0.093607140	2 4, -4
23, 24	0.000382540	0.094372220	1 7, -7
25, 26	0.000382130	0.095136480	3 2, -2
27	0.000373790	0.095510270	3 0
28, 29	0.000262110	0.096034490	2 5, -5
30, 31	0.000251910	0.096538310	1 8, -8
32, 33	0.000224370	0.096987050	3 3, -3
34, 35	0.000215100	0.097417250	3 1, -1
36, 37	0.000173860	0.097764970	1 9, -9
38, 39	0.000172040	0.098109050	2 6, -6
40, 41	0.000143750	0.098396550	3 4, -4
42, 43	0.000133880	0.098664310	4 2, -2
44	0.000129880	0.098794190	4 0
45, 46	0.000124560	0.099043310	1 10, -10
47, 48	0.000118870	0.099281050	2 7, -7
49, 50	0.000097881	0.099476812	3 5, -5
51, 52	0.000091963	0.099660738	1 11, -11
53, 54	0.000089079	0.099838896	4 3, -3
55, 56	0.000085452	0.100009800	2 8, -8
57, 58	0.000084808	0.100179416	4 1, -1
59, 60	0.000069736	0.100318888	3 6, -6
61, 62	0.000069646	0.100458180	1 12, -12
63, 64	0.000063398	0.100584976	2 9, -9
65, 66	0.000062420	0.100709816	4 4, -4
67, 68	0.000058784	0.100827384	5 2, -2
69	0.000057899	0.100885283	5 0
70, 71	0.000053855	0.100992993	1 13, -13
72, 73	0.000051459	0.101095911	3 7, -7
74, 75	0.000048271	0.101192453	2 10, -10
76, 77	0.000045524	0.101283501	4 5, -5
78, 79	0.000042494	0.101368489	5 3, -3
80, 81	0.000042428	0.101453345	1 14, -14
82, 83	0.000041356	0.101536057	5 1, -1
84, 85	0.000039058	0.101614173	3 8, -8
86, 87	0.000037553	0.101689279	2 11, -11
88, 89	0.000034265	0.101757809	4 6, -6
90, 91	0.000033936	0.101825681	1 15, -15
92, 93	0.000031680	0.101889041	5 4, -4
94	0.000030739	0.101919780	6 0
95, 96	0.000030339	0.101980458	3 9, -9
97, 98	0.000029757	0.102039972	2 12, -12
99, 100	0.000029734	0.102099440	6 2, -2

$$\Delta^2 = (\frac{1}{4}\pi D^2)^{-1} \sum_{n,n'} \beta_n^* \beta_{n'} \int dr W(r, D) f_n^*(r) f_{n'}(r). \quad (39)$$

Making use of the orthonormality condition expressed in Eq. (9), Eq. (39) can be reduced to the form

$$\Delta^2 = (\frac{1}{4}\pi D^2)^{-1} \sum_n \beta_n^* \beta_n = \sum_n S_n^2, \quad (40)$$

where S_n is defined in terms of β_n as

$$S_n = [(\frac{1}{4}\pi D^2)^{-1} \beta_n^* \beta_n]^{1/2}. \quad (41)$$

Since β_n is a Gaussian random variable, so is S_n . Moreover, since β_n has a zero mean value, so does S_n . Its variance is

$$\sigma_n^2 = \langle S_n^2 \rangle. \quad (42)$$

Substituting Eq. (41) into Eq. (42), and making use of Eq. (12), we obtain

$$\sigma_n^2 = (\frac{1}{4}\pi D^2)^{-1} \langle \beta_n^* \beta_n \rangle = (\frac{1}{4}\pi D^2)^{-1} B_n^2(D). \quad (43)$$

Now if we make use of Eq. (23), we can cast Eq. (43) in the form

$$\sigma_n^2 = (4/\pi)(D/r_0)^{5/3} \mathfrak{B}_n^2. \quad (44)$$

At this point, we see that the aperture averaged, squared wave-front distortion Δ^2 as given in Eq. (40), is equal to the sum of the square of a set of Gaussian random variables S_n , each with mean value zero, and with a variance σ_n^2 given by Eq. (44). The value of σ_n^2 is given in terms of the eigenvalues we have just solved for and which are listed in Table I. The dependence of the aperture-averaged wave-front distortion squared on aperture diameter D and on the wave-front distortion coherence length r_0 is contained in the $(D/r_0)^{5/3}$ dependence shown in Eq. (44).

Our problem now is to figure out the probability that the random variables S_n , squared and summed, as in the right-hand side of Eq. (40), will have a value less than some amount. We define a good image as one that is formed when the wave-front distortion over the aperture is less than 1 rad rms, i.e., when Δ^2 is less than 1 rad.² The probability of obtaining such a good short-exposure image

$$\begin{aligned} &\text{Prob}(\text{good short-exposure image}) \\ &= \text{Prob}\left(\sum_n S_n^2 \leq 1 \text{ rad}^2\right) \end{aligned} \quad (45)$$

can be written in terms of the Gaussian distributions for each of the variables

$$\begin{aligned} &\text{Prob}(\text{good short-exposure image}) \\ &= \prod_{n=1}^{\infty} \int_{\text{limits}} dS_n (2\pi \sigma_n^2)^{-1/2} \\ &\quad \times \exp(-\frac{1}{2} S_n^2 / \sigma_n^2). \end{aligned} \quad (46)$$

where the limits correspond to a hypersphere in the multidimensional S_n space, the sphere being of unit radius.

This calculation, though it bears some resemblance to the ordinary chi-squared integral, because of the depen-

TABLE II. Probability of obtaining a good short-exposure image.

D/r_0	Probability
2	0.986 ± 0.006
3	0.765 ± 0.005
4	0.334 ± 0.014
5	$(9.38 \pm 0.33) \times 10^{-2}$
6	$(1.915 \pm 0.084) \times 10^{-2}$
7	$(2.87 \pm 0.57) \times 10^{-3}$
10	$(1.07 \pm 0.48) \times 10^{-6}$
15	$(3.40 \pm 0.59) \times 10^{-15}$

dence of σ_n^2 on n , can not be performed in closed form. We have therefore made use of Monte Carlo techniques for evaluation of this probability. The results obtained are shown in Table II.

In Fig. 1, we have plotted this probability in a form which makes manifest a negative exponential dependence on $(D/r_0)^2$. As can be seen, the results are very strongly suggestive of this type of dependence, and we have fitted the equation

$$\text{Prob} \approx 5.6 \exp[-0.1557 (D/r_0)^2] \quad (\text{if } D/r_0 \geq 3.5) \quad (47)$$

to the data. It is quite obvious that the probability is a strong function of D/r_0 , and that if we intend to have a reasonable probability of obtaining a good short-exposure image, we must be careful not to push the aperture diameter beyond some reasonable multiple of r_0 . Ex-

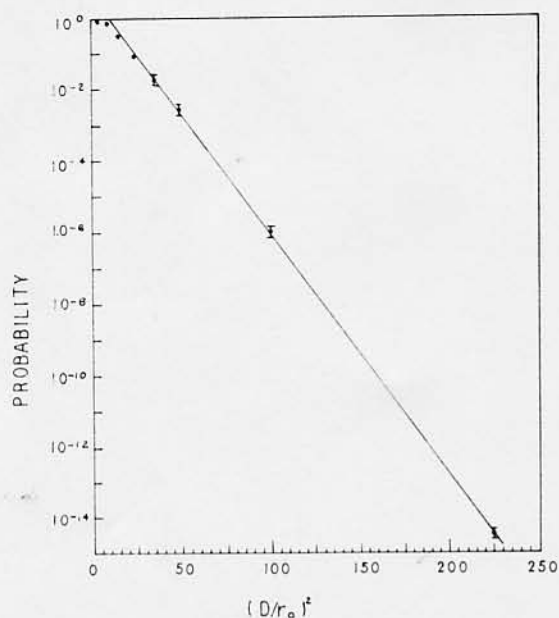


FIG. 1. Probability of obtaining a good short-exposure image. The probability is plotted on a logarithmic scale against the aperture diameter divided by r_0 ratio squared. A straight line on this graph shows an exponential dependence of the probability on the aperture area. The data plotted corresponds to the values in Table II, with the spread due to the fact that Monte Carlo integral evaluation was used. The straight-line fit to the data matches Eq. (47).

actly how large this multiple can be depends on how many short-exposure images we are willing to take or how long we are willing to wait before we get a good image.

VI. DISCUSSION OF RESULTS

The results presented in Table II and Fig. 1 are fairly self-explanatory. They are certainly in good agreement with Hufnagel's¹ conjecture that the probability ought to be a negative exponential function of aperture area. The quantitative results tell us that if we want a probability of the order of 1×10^{-3} of obtaining a good short-exposure image, the aperture diameter should not be significantly larger than $7r_0$ or $8r_0$. If we are willing to accept a probability of 1×10^{-6} , only a minor increase in diameter up to $10r_0$ is allowed. Working with diameters any larger than this leads to what are almost certainly unacceptable probabilities.

It is particularly interesting to note the operationally high probabilities that apply for diameters of about $7r_0$ or a bit less. This suggests that for many imagery purposes significantly better than ordinary turbulence-limited resolution can be achieved if we carefully choose our aperture diameter and take several hundred short-exposures and select from these the best. This, of course, calls for not merely an adjustable aperture on our imaging system, but some method of measuring r_0 so that we know what aperture diameter to use.

It is appropriate to note that the probability we have calculated applies independently to separate isoplanatic patches on the image. This means that in any one image, rather than its being entirely good or entirely poor resolution, there will be distributed over the image field-of-view a set of rather small regions, isoplanatic patches, in which the resolution is good. The rest of the image area will have much poorer resolution. To image a large object and determine all of the fine details of that object, it would be necessary to piece together the image from a set of short-exposures, selecting the high-resolution regions in each of the images, to put together one high-resolution image.

The assumption that we are dealing with a short-exposure image is basically equivalent to the assumption that the random wave-front tilt does not change significantly during the exposure. So long as the wave-front tilt is constant, it does not affect the sharpness of the image—but if the exposure period is long enough to allow any significant wave-front tilt change, this can smear the image and degrade the resolution. We have recently carried out calculations of the rate of wave-front tilt change based on the work of Greenwood and Fried,⁵ and from this have been able to estimate the allowable exposure time for short-exposure imagery of the type we have been considering here. We find that if along the propagation path there is a uniform wind speed V perpendicular to the path, then the change in wave-front tilt will be less than $\frac{1}{2}\lambda/D$ if the exposure time is less than $\tau = \frac{1}{2}r_0/V$. We believe this is a proper criterion for selecting the short-exposure period in seeking an

accidental occurrence of near diffraction-limited imagery through turbulence.

As a word of caution in interpreting and applying our results, it is appropriate to recollect here that we have assumed that the statistics of turbulence and the value of r_0 remain constant over the period of the "experiment." What we have calculated is the probability that a sample drawn from the ensemble of wave-front distortions will be a sample with almost no distortion. Our ensemble is restricted to samples from the same value of r_0 , i. e., there is no change in the strength of turbulence over the propagation path during the experiment. There is, however, a larger ensemble associated with different values of r_0 . r_0 changes with changing turbulence conditions, and this larger ensemble includes cases for which r_0 may sometimes, though infrequently, be significantly larger than the typical value. Average probabilities over this grand ensemble might be expected to yield significantly higher probabilities of a good short-exposure image for large aperture diameters. However, obtaining an average over this grand ensemble corresponds in the physical world to waiting over periods of hours, perhaps days, for the variety of turbulence conditions that can exist over the propagation path. This may not be practical. The ensemble average for which we have evaluated the probability of obtaining a good

short-exposure image in this paper corresponds to taking a set of images in rapid succession over a short period of time during which we do not expect or require the nature of the turbulence in the propagation path to change. We believe this case corresponds to a variety of nominal practical limitations. Our results show that in this type of situation, a judicious choice of aperture diameter and careful selection of the images to be utilized can yield results which have significantly better image resolution than the typical atmospheric turbulence limit.

*Originally presented at the NATO AGARD Electromagnetic Panel meeting on "Optical Propagation in the Atmosphere," held in Lyngby, Denmark, October 1975, with the title, "How Many Pictures Do You Have To Take To Get A Good One?"

¹R. E. Hufnagel, in *Restoration of Atmospherically Degraded Images* (National Academy of Sciences, Washington, D. C., 1966), Vol. 3, Appendix 2, p. 11.

²D. L. Fried, "Optical heterodyne detection of an atmospherically distorted signal wavefront," *Proc. IEEE* 55, 57 (1967).

³D. L. Fried, "Statistics of a geometric representation of wave-front distortion," *J. Opt. Soc. Am.* 55, 1427 (1965).

⁴W. I. Tatarski, *Wave Propagation in a Turbulent Medium* (McGraw-Hill, New York, 1961).

⁵D. P. Greenwood and D. L. Fried, "Power spectra requirements for wave-front compensative systems," *J. Opt. Soc. Am.* 66, 193 (1976).