Waves in plasma

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This is a short introduction on waves in a non-relativistic plasma. We will consider a plasma of electrons and protons which is fully ionized, non-relativistic and homogeneous.

1 Preliminaries

1.1 Maxwell's equations

The electromagnetic field $(\vec{E}, \vec{B})$, in such a plasma, is governed by Maxwell's equations:

$$\vec{\nabla} \cdot \vec{E} = \frac{e\,(n_p - n_e)}{\epsilon_0}\,, \quad (1)$$

$$\vec{\nabla} \cdot \vec{B} = 0\,, \quad (2)$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}\,, \quad (3)$$

$$\vec{\nabla} \times \vec{B} = \mu_0\,\vec{J} + \frac{1}{c^2}\,\frac{\partial \vec{E}}{\partial t}\,, \quad (4)$$

where n_p and n_e are respectively the local proton density and the local electron density. In our case, we can add the definition of the total current density vector

$$\vec{J} \equiv \vec{J}_e + \vec{J}_p = e\,(n_p\,\vec{V}_p - n_e\,\vec{V}_e)\,, \quad (5)$$

where $\vec{V}_p$ and $\vec{V}_e$ are respectively the local proton fluid velocity and the local electron fluid velocity (see Sec. 1.2).

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1.2 Fluid reduction

For a distribution function $f(\vec{r},\vec{v},t)$ $(= f(\vec{x},t))$, the Fokker-Planck equation is such that

$$\frac{\partial f}{\partial t} = -\frac{\partial}{\partial x_i}\left[\frac{< \Delta x_i >}{\Delta t} f - \frac{\partial}{\partial x_j}\left(\frac{< \Delta x_i \, \Delta x_j >}{2\Delta t}\right) f\right] . \tag{6}$$

For a process on a time Δt much shorter than the characteristic colliding time, we can neglect the diffusion terms, and we obtain

$$\frac{\partial f}{\partial t} = -\frac{\partial}{\partial \vec{r}}\left(\frac{< \Delta \vec{r} >}{\Delta t} f\right) - \frac{\partial}{\partial \vec{v}}\left(\frac{< \Delta \vec{v} >}{\Delta t} f\right) . \tag{7}$$

The plasma dynamics is dominated by the electromagnetic field and we have

$$\frac{< \Delta \vec{r} >}{\Delta t} = \vec{v} , \tag{8}$$

$$\frac{< \Delta \vec{v} >}{\Delta t} = \frac{q}{m} \left(\vec{E}(\vec{r},t) + \vec{v} \times \vec{B}(\vec{r},t)\right) . \tag{9}$$

We then deduce the Vlasov's equation $(\frac{\partial}{\partial \vec{v}} \times \vec{v} = 0)$

$$\frac{\partial f}{\partial t} + \vec{v}\, \frac{\partial f}{\partial \vec{r}} + \frac{q}{m} \left(\vec{E}(\vec{r},t) + \vec{v} \times \vec{B}(\vec{r},t)\right) \frac{\partial f}{\partial \vec{v}} = 0 , \tag{10}$$

and the hydrodynamics quantities $(\mathrm{d}\vec{v} = \mathrm{d}v_1 \mathrm{d}v_2 \mathrm{d}v_3)$:

$$n(\vec{r},t) \equiv \int f(\vec{r},\vec{v},t)\, \mathrm{d}\vec{v} , \tag{11}$$

$$\vec{V}(\vec{r},t) \equiv \frac{\int \vec{v}\, f(\vec{r},\vec{v},t)\, \mathrm{d}\vec{v}}{n(\vec{r},t)} , \tag{12}$$

$$\bar{\bar{P}}(\vec{r},t) \equiv \int m\, (\vec{v} - \vec{V}) \otimes (\vec{v} - \vec{V})\, f(\vec{r},\vec{v},t)\, \mathrm{d}\vec{v} , \tag{13}$$

respectively the density, the Euler's velocity (or fluid velocity) and the kinetic pressure tensor.
The integration of the Vlasov's equation yields the continuity equation

$$\frac{\partial n}{\partial t} = -\vec{\nabla} \cdot (n\, \vec{V}) , \tag{14}$$

because the velocity distribution at the infinity equals to 0,

$$\int \left(\vec{E} + \vec{v} \times \vec{B}\right) \frac{\partial f}{\partial \vec{v}} \, \mathrm{d}\vec{v} = \int \frac{\partial}{\partial \vec{v}} \left[\left(\vec{E} + \vec{v} \times \vec{B}\right) f\right] \mathrm{d}\vec{v} = 0 \,. \tag{15}$$

The Euler's equation is obtained by the first order moment of the Vlasov's equation,

$$\int \vec{v} \left(\frac{\partial f}{\partial t} + \vec{v} \, \frac{\partial f}{\partial \vec{r}} + \frac{q}{m} \left(\vec{E} + \vec{v} \times \vec{B}\right) \frac{\partial f}{\partial \vec{v}}\right) \mathrm{d}\vec{v} = 0 \,, \tag{16}$$

and, because $\vec{v} \otimes \vec{v} = (\vec{v} - \vec{V}) \otimes (\vec{v} - \vec{V}) - \vec{V} \otimes \vec{V} + \vec{v} \otimes \vec{V} + \vec{V} \otimes \vec{v}$,

$$\int \vec{v} \left(\vec{v} \, \frac{\partial f}{\partial \vec{r}}\right) \mathrm{d}\vec{v} = \int (\vec{v} \otimes \vec{v}) \, \frac{\partial f}{\partial \vec{r}} \, \mathrm{d}\vec{v} = \vec{\nabla} \cdot \left(\frac{\bar{\bar{P}}}{m} + n \, \vec{V} \otimes \vec{V}\right) . \tag{17}$$

Also, we have, by integration parts,

$$\int \vec{v} \, \frac{\partial}{\partial \vec{v}} \left[\left(\vec{E} + \vec{v} \times \vec{B}\right) f\right] \mathrm{d}\vec{v} = -n \left(\vec{E} + \vec{V} \times \vec{B}\right) . \tag{18}$$

Then, the Euler's equation is

$$\frac{\partial \vec{V}}{\partial t} + \left(\vec{V} \cdot \vec{\nabla}\right) \vec{V} = -\frac{\vec{\nabla} \bar{\bar{P}}}{n \, m} + \frac{q}{m} \left(\vec{E} + \vec{V} \times \vec{B}\right) . \tag{19}$$

With an isotropic and adiabatic assumption: $\bar{\bar{P}} = P \, \mathbf{I}$ and $P \, n^{-\gamma} = constant$, where $\gamma = c_p / c_v$ is the adiabatic index.
In general anisotropic case, by writing $\bar{\bar{P}} = P_\perp \, I + (P_\parallel - P_\perp) \, \frac{\vec{B} \otimes \vec{B}}{B^2}$, and in adiabatic transformations (of duration $\Delta t \gg 1/\omega_{ce}$), we obtain

$$\frac{P_\perp}{n \, B} = constant \,, \tag{20}$$

$$\frac{P_\parallel \, B^2}{n^3} = constant \,. \tag{21}$$

In Eq. (19), for a two fluid model (p-e^- plasma), we could have to add a term of momentum exchange which results from taking account a collision term in the Fokker-Planck equation (6): it can be approximated by $\nu \, \Delta\vec{v}$, where ν

is the colliding frequency or the momentum exchange rate per collision[1]. It yields two coupled Euler's equations:

$$\frac{\partial \vec{V}_e}{\partial t} + \left(\vec{V}_e \cdot \vec{\nabla}\right) \vec{V}_e = -\frac{\vec{\nabla} \bar{\bar{P}}_e}{n_e\, m_e} - \frac{e}{m_e} \left(\vec{E} + \vec{V}_e \times \vec{B}\right) - \nu_{ep}\,(\vec{V}_e - \vec{V}_p)\,, \quad (22)$$

$$\frac{\partial \vec{V}_p}{\partial t} + \left(\vec{V}_p \cdot \vec{\nabla}\right) \vec{V}_p = -\frac{\vec{\nabla} \bar{\bar{P}}_p}{n_p\, m_p} + \frac{e}{m_p} \left(\vec{E} + \vec{V}_p \times \vec{B}\right) - \nu_{pe}\,(\vec{V}_p - \vec{V}_e)\,, \quad (23)$$

where $\nu_{ep} \simeq n_p\, \sigma_{ep}\, |v_e - v_p|$ and $\nu_{pe} \simeq n_e\, \sigma_{pe}\, |v_p - v_e|$.

2 Magneto-Hydro-Dynamics

2.1 MHD model assumptions

We want to construct a single fluid theory:
(1) We consider phenomena that occur on a scale time τ_0 such that $\tau_0\, \omega_{cp} \gg 1$, where the Larmor's pulsation $\omega_{cp} = e\, B/m_p$, or on a lenght scale ℓ_0 such that $\ell_0\, \omega_{pp} \gg c$, where the plasma pulsation $\omega_{pp} = \sqrt{n_p\, e^2/\epsilon_0\, m_p}$. We will define the Alfvén velocity as $V_A = (\omega_{cp}/\omega_{pp})\, c \ll c$.
(2) We assume the local thermodynamics equilibrium[2] ($T_e \simeq T_p$), and a quasi-neutral p-e^- plasma: $|n_p - n_e| \ll n_p + n_e$.
(3) $\bar{\bar{P}} = P\, \mathbf{I}$ and $P\, n^{-\gamma} = \mathit{constant}$ (isotropic and adiabatic assumption).
(4) We define the following quantities:

$$\rho \equiv n_p\, m_p + n_e\, m_e \simeq n_p\, m_p\,, \quad (24)$$

$$\vec{V} \equiv \frac{n_p\, m_p\, \vec{V}_p + n_e\, m_e\, \vec{V}_e}{n_p\, m_p + n_e\, m_e} \simeq \vec{V}_p\,, \quad (25)$$

$$P \equiv P_p + P_e\,, \quad (26)$$

$$\vec{J} \equiv e\,(n_p\, \vec{V}_p - n_e\, \vec{V}_e)\,. \quad (27)$$

With these assumptions, the Eq. (22) and (23) lead to

$$\rho \left[\frac{\partial \vec{V}}{\partial t} + \left(\vec{V} \cdot \vec{\nabla}\right) \vec{V}\right] = -\vec{\nabla} P + (n_p - n_e)\, e\, \vec{E} + \vec{J} \times \vec{B} + (\nu_{ep}\, n_e\, m_e - \nu_{pe}\, n_p\, m_p)\,(\vec{V}_e - \vec{V}_p)\,,$$

[1] It depends from the interaction modelling between the two particle species.
[2] We have the approximation $< \vec{v}_e^{\,2} > / < \vec{v}_p^{\,2} > \simeq m_p/m_e$, but $|\vec{V}_e - \vec{V}_p| \simeq 0$.

or, taking into account the quasi-neutrality assumption and equality in momentum exchange ($n_e\, m_e \nu_{ep} = n_p\, m_p\, \nu_{pe}$ or $\nu_{ep} = (m_p/m_e)\, \nu_{pe}$),

$$\rho \left[\frac{\partial \vec{V}}{\partial t} + \left(\vec{V} \cdot \vec{\nabla} \right) \vec{V} \right] = -\vec{\nabla} P + \vec{J} \times \vec{B} \,. \tag{28}$$

Also, because $\vec{V_e} = \vec{V_p} - \vec{J}/e\, n_e \simeq \vec{V} - \vec{J}/e\, n_e$, we deduce from Eq. (22) that

$$\vec{E} + \vec{V} \times \vec{B} - \frac{m_e\, \nu}{n_e\, e^2}\, \vec{J} - \frac{\vec{J} \times \vec{B}}{n_e\, e} = 0 \,, \tag{29}$$

by neglecting $\mathrm{d}\vec{V_e}/\mathrm{d}t$ and $\vec{\nabla} P_e$ in the fluid model, and with $\nu = \nu_{ep}$.
With $\eta \equiv \frac{n_e\, e^2}{m_e\, \nu}$ (Spitzer conductivity), $\omega_{ce} \equiv e\, B/m_e$ and $\vec{b} = \vec{B}/B$, the generalized Ohm law is written

$$\vec{J} + \frac{\omega_{ce}}{\nu} \left(\vec{J} \times \vec{b} \right) = \eta \left(\vec{E} + \vec{V} \times \vec{B} \right) \,. \tag{30}$$

The current density vector, with $\vec{E}_{\parallel} = (\vec{E} \cdot \vec{b})\, \vec{b}$ and $\vec{E}_{\perp} = \vec{b} \times (\vec{E} \times \vec{b})$, can be easily expressed

$$\vec{J} = \eta\, \vec{E}_{\parallel} + \eta_{\perp} \left(\vec{E}_{\perp} + \vec{V} \times \vec{B} \right) + \eta_{\times}\, \vec{b} \times \left(\vec{E}_{\perp} + \vec{V} \times \vec{B} \right) \,, \tag{31}$$

where $\eta_{\perp} = \eta/(1 + \omega_c^2/\nu^2)$ and $\eta_{\times} = \eta/(\nu/\omega_c + \omega_c/\nu)$.

There are two asymptotic limits:
(1) When $\omega_{ce} \ll \nu$, the Ohm's law becomes isotropic and $\vec{J} = \eta \left(\vec{E} + \vec{V} \times \vec{B} \right)$. This is the "resistive" MHD regime.
(2) When the plasma is collisionless, i.e $\nu \to 0$ (or $\eta \to +\infty$), with a finite ratio ω_{ce}/ν, we have the approximation of the "ideal MHD" or the non-resistive MHD regime: $\vec{E} + \vec{V} \times \vec{B} = 0$. But, when $|\vec{J} \times \vec{B}|/e\, n_e\, |\vec{V} \times \vec{B}| \ll 1$, we have also $\vec{J} = \eta \left(\vec{E} + \vec{V} \times \vec{B} \right)$: this is the case when $\ell_0\, \omega_{pp} \gg c$ (first MHD assumption) and $V_0 \sim \ell_0/\tau_0 \neq 0$. Otherwise it is necessary to have $\omega_{ce} \ll \nu$.

When the electromagnetic field is slightly variable (over scales ℓ_0 and τ_0), $|\vec{\nabla} \times \vec{E}| \sim E_0/\ell_0 \sim B_0/\tau_0$, $|\partial \vec{E}/\partial t| \sim E_0/\tau_0$ and $|\vec{\nabla} \times \vec{B}| \sim B_0/\ell_0$, where E_0 and B_0 are the characteristic values of E and B. Thus we have

$$\frac{|(1/c^2)\, \partial \vec{E}/\partial t|}{|\vec{\nabla} \times \vec{B}|} \sim \frac{(\ell_0/\tau_0)^2}{c^2} \ll 1 \,, \tag{32}$$

in the non-relativistic MHD regime, and the Eq. (4) becomes $\vec{\nabla} \times \vec{B} = \mu_0 \, \vec{J}$. Assuming an isotropic Ohm's law, we deduce from Eq. (3) and (4),

$$\frac{\partial \vec{B}}{\partial t} - \vec{\nabla} \times (\vec{V} \times \vec{B}) = \nu_m \, \Delta \vec{B} \, , \tag{33}$$

where $\nu_m = 1/\mu_0 \, \eta$ is the magnetic diffusivity. We can form the ratio between the advective and the diffusive terms

$$\frac{|\vec{\nabla} \times (\vec{V} \times \vec{B})|}{|\nu_m \, \Delta \vec{B}|} \sim \frac{V_0 \, B_0/\ell_0}{\nu_m \, B_0/\ell_0^2} = \frac{V_0 \, \ell_0}{\nu_m} \, , \tag{34}$$

where V_0 ($\ll c$) is the characteristic velocity ($\sim V_A$) of the MHD fluid. This ratio is defined as the magnetic Reynold number $\mathcal{R}_m$. When $\mathcal{R}_m \lesssim 1$, the plasma dynamics is dominated by its resistivity and its diffusion. When $\mathcal{R}_m \gg 1$, this is the ideal MHD regime: the diffusion can be neglected and, the magnetic field and the MHD fluid are "frozen" together.

2.2 Ideal MHD and waves

According to the previous section, the ideal MHD equations are:

$$\frac{\partial \rho}{\partial t} = -\vec{\nabla} \left(\rho \, \vec{V} \right) \, , \tag{35}$$

$$\frac{\partial \vec{B}}{\partial t} = \vec{\nabla} \times (\vec{V} \times \vec{B}) \, , \tag{36}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(P \, \rho^{-\gamma} \right) = 0 \, , \tag{37}$$

$$\rho \left[\frac{\partial \vec{V}}{\partial t} + \left(\vec{V} \cdot \vec{\nabla} \right) \vec{V} \right] = -\vec{\nabla} P + \frac{\vec{\nabla} \times \vec{B}}{\mu_0} \times \vec{B} \, . \tag{38}$$

In the Eq. (38), the term $(\vec{\nabla} \times \vec{B}) \times \vec{B}$ can be written $(\vec{B} \cdot \vec{\nabla}) \, \vec{B} - (1/2) \, \vec{\nabla} B^2$. Thus, we have

$$\rho \left[\frac{\partial \vec{V}}{\partial t} + \left(\vec{V} \cdot \vec{\nabla} \right) \vec{V} \right] = -\vec{\nabla} P - \vec{\nabla} P_m + \vec{T}_m \, , \tag{39}$$

where the magnetic pressure $P_m \equiv B^2/2\mu_0$, and the magnetic tension, $\vec{T}_m$, is such that

$$\vec{T}_m \equiv \frac{(\vec{B} \cdot \vec{\nabla}) \, \vec{B}}{\mu_0} = \frac{\mathrm{d}}{\mathrm{d}s} \left(\frac{B^2}{2\mu_0} \right) \vec{b} + \frac{B^2}{\mu_0} \, \frac{\vec{b}_\perp}{\mathcal{R}_c} \, , \tag{40}$$

where s is the curvilinear abscissa, $\vec{b}_\perp$ is the unit vector perpendicular to $\vec{b} = \vec{B}/B$, and $\mathcal{R}_c$, the curvature radius of the magnetic field line. Thus, the Laplace's force (per unit of volum) can be written

$$\vec{f}_L = \vec{J} \times \vec{B} = -\vec{\nabla}_\perp P_m + \frac{2P_m}{\mathcal{R}_c}\,\vec{b}_\perp\,. \tag{41}$$

If $P/P_m < 1$, the plasma undergoes the magnetic dynamics, and if $P/P_m > 1$, the magnetic field undergoes the plasma dynamics.
At last, the Eq. (37) can be transformed

$$\frac{\partial P}{\partial t} = -\vec{v}\,\vec{\nabla} P - \gamma\, P\,\vec{\nabla}\vec{v}\,. \tag{42}$$

Let us study the effects of a weak perturbation. The linearization process is the following one: considering a Fourier decomposition, all complex quantities X will be such that $X = X_0 + x$ where X_0 is a real time invariant quantity and $|x/X_0| \ll 1$. The perturbative quantity x will be $\propto \exp[i\,(\omega\,t - \vec{k}\cdot\vec{r})]$ (where $i^2 = -1$). The only condition on the pulsation ω is that $\omega \ll \omega_{cp}$. Thus, we will have $\partial/\partial t \equiv i\,\omega$, $\vec{\nabla}\cdot \equiv -i\,\vec{k}$, $\vec{\nabla}\times \equiv -i\,\vec{k}\times$ and $\vec{\nabla}^2 \equiv -k^2$, when one derive these perturbative quantities (only).
Assuming that $\vec{V}_0 = \vec{0}$ and writing $\rho = \rho_0 + \delta\rho$, the Eqs. (35) to (38) yield

$$\begin{aligned}
\delta\rho &= \frac{\rho_0}{\omega}\,\vec{k}\cdot\vec{v}\,, && (43)\\
\vec{b} &= -\frac{1}{\omega}\,\vec{k}\times(\vec{v}\times\vec{B}_0)\,, && (44)\\
i\omega\, p &= -\vec{v}\,\vec{\nabla} P_0 + i\gamma\, P_0\,\vec{k}\cdot\vec{v}\,, && (45)\\
i\omega\,\rho_0\,\vec{v} &= -\vec{\nabla}\left[p + \frac{\vec{B}_0\cdot\vec{b}}{\mu_0}\right] + \frac{1}{\mu_0}\left[(\vec{B}_0\cdot\vec{\nabla})\,\vec{b} + (\vec{b}\cdot\vec{\nabla})\,\vec{B}_0\right]\,. && (46)
\end{aligned}$$

In fact, the Eq. (46) is obtained because, following the Eq. (39), we have

$$\vec{\nabla} P_0 = -\vec{\nabla}\left(\frac{B_0^2}{2\mu_0}\right) + \frac{(\vec{B}_0\cdot\vec{\nabla})\,\vec{B}_0}{\mu_0}\,. \tag{47}$$

2.3 Example: MHD waves in homogeneous medium

We consider here a simple case: the medium is homogeneous i.e the spatial variation of quantities X_0 are negligible. The Eqs. (45) and (46) yield

$$\omega\, p = \gamma\, P_0\, \vec{k}\cdot\vec{v}\,, \tag{48}$$

$$\omega\, \rho_0\, \vec{v} = p\,\vec{k} + \frac{1}{\mu_0}\,\vec{B}_0\times\left(\vec{k}\times\vec{b}\right)\,. \tag{49}$$

With the Eq. (44), we obtain

$$\omega^2\,\rho_0\,\vec{v} = \gamma\,P_0\,\left(\vec{k}\cdot\vec{v}\right)\,\vec{k} - \frac{1}{\mu_0}\,\vec{B}_0\times\left[\vec{k}\times\left(\vec{k}\times\left(\vec{v}\times\vec{B}_0\right)\right)\right]\,. \tag{50}$$

Let us define the basis vectors $\vec{e}_\parallel = \vec{B}_0/|\vec{B}_0|$, $\vec{e}_\perp = \vec{k}_\perp/|\vec{k}_\perp|$ when $|\vec{k}_\perp| \neq 0$ (otherwise $\vec{e}_\perp$ is an arbitrary perpendicular unit vector), and $\vec{e}_\times = \vec{e}_\parallel\times\vec{e}_\perp$, with $\vec{k} = k_\parallel\,\vec{e}_\parallel + k_\perp\,\vec{e}_\perp$ and $\vec{v} = v_\parallel\,\vec{e}_\parallel + v_\perp\,\vec{e}_\perp + v_\times\,\vec{e}_\times$. We will have

$$-\frac{1}{\mu_0}\,\vec{B}_0\times\left[\vec{k}\times\left(\vec{k}\times\left(\vec{v}\times\vec{B}_0\right)\right)\right] = \frac{1}{\mu_0}\,B_0^2\left(v_\perp\,k^2\,\vec{e}_\perp + v_\times\,k_\parallel^2\,\vec{e}_\times\right)\,. \tag{51}$$

Thus, the Eq. (50) leads to the system

$$\left(\omega^2 - C_s^2\,k_\parallel^2\right)\,v_\parallel - C_s^2\,k_\parallel\,k_\perp\,v_\perp = 0\,, \tag{52}$$

$$\left(\omega^2 - C_s^2\,k_\perp^2 - V_A^2\,k^2\right)\,v_\perp - C_s^2\,k_\parallel\,k_\perp\,v_\parallel = 0\,, \tag{53}$$

$$\left(\omega^2 - V_A^2\,k_\parallel^2\right)\,v_\times = 0\,, \tag{54}$$

with the sound velocity $C_s = \sqrt{\gamma\,P_0/\rho_0}$ and the Alfvén velocity $V_A = \sqrt{B_0^2/(\rho_0\mu_0)}$ for $(B_0^2/\mu_0)/\rho_0\,c^2 \ll 1$. In its matricial form, this system becomes

$$\begin{pmatrix} (\omega^2 - C_s^2\,k_\parallel^2) & -C_s^2\,k_\parallel\,k_\perp & 0 \\ -C_s^2\,k_\parallel\,k_\perp & (\omega^2 - C_s^2\,k_\perp^2 - V_A^2\,k^2) & 0 \\ 0 & 0 & (\omega^2 - V_A^2\,k_\parallel^2) \end{pmatrix}\begin{pmatrix} v_\parallel \\ v_\perp \\ v_\times \end{pmatrix} = 0\,. \tag{55}$$

The solutions of this system (proper modes) are determined by det = 0. The Alfvén waves are then defined by the following dispersion relation,

$$\omega^2 = V_A^2\,k_\parallel^2\,, \tag{56}$$

and the fast (+) and slow (−) magnetosonic waves are determined by the dispersion relation,

$$\omega_{f,s}^2 = \frac{k^2}{2}\left[(C_s^2 + V_A^2) \pm \sqrt{(C_s^2 + V_A^2)^2 - 4\, C_s^2 V_A^2 \left(\frac{k_\parallel}{k}\right)^2}\,\right] . \qquad (57)$$

3 Electromagnetic waves in fluid model

3.1 Assumptions

In the Sec. 3, we will consider the following assumptions:

General assumption of adiabaticity:
(1) The phase velocity of waves, $v_\phi = \omega/k$, is supposed to be $\gg v_{th}$, the thermal velocity of charged particle species.
(2) The energy density of the EM waves is $\ll$ the thermal energy density.
(3) The damping of EM waves is neglected.
(4) The collision effects are neglected: $\nu \to 0$.

Additional assumption: the plasma is homogeneous.

We will define:
The plasma pulsation of the species a: $\omega_{pa} = \sqrt{n_a\, q_a^2/\epsilon_0\, m_a}$, where n_a, q_a and m_a, are respectively the density, the charge and the mass of a.
The cyclotronic or Larmor pulsation of the species a: $\omega_{ca} = |q_a|\, B_0/m_a$, where B_0 is the intensity of the magnetic field.

3.2 Electromagnetic mode

The plasma is assumed to be isotropic and **unmagnetized** ($B_0 = 0$). We first neglect the ionic response ($\vec{J}_p$) and the temperature ($T_e = T_p \simeq 0$). With the previous assumptions, the Eq. (22) leads to

$$\frac{\partial \vec{V}_e}{\partial t} + \left(\vec{V}_e \cdot \vec{\nabla}\right) \vec{V}_e = -\frac{e}{m_e}\left(\vec{E} + \vec{V}_e \times \vec{B}\right) . \qquad (58)$$

The fields $\vec{E}$ and $\vec{B}$ are considered as perturbations. Following the linearization process, as explained in Sec. 2.2, the Eq. (58) simply becomes

$$\frac{\partial \vec{V}_e}{\partial t} = -\frac{e}{m_e}\,\vec{E}\,, \tag{59}$$

and, because $\vec{J}_e = n_e\, e\, \vec{V}_e$, we obtain

$$\vec{J}_e = -i\,\frac{\omega_{pe}^2}{\omega}\,\epsilon_0\,\vec{E}\,. \tag{60}$$

Thus, the Eqs. (3) and (4) could be written

$$\vec{k}\times\vec{E} = \omega\,\vec{B}\,, \tag{61}$$

$$\vec{k}\times\vec{B} = i\,\vec{J}_e - \omega\,\epsilon_0\,\vec{E}\,, \tag{62}$$

and we deduce

$$\left(\vec{k}\cdot\vec{E}\right)\,\vec{k} - k^2\,\vec{E} = \frac{\omega_{pe}^2-\omega^2}{c^2}\,\vec{E}\,. \tag{63}$$

For the *electromagnetic mode* ($\vec{k}\cdot\vec{E}=0$), this equation yields the following dispersion relation:

$$\omega^2 = \omega_{pe}^2 + k^2\,c^2\,. \tag{64}$$

For $\omega > \omega_{pe}$, the phase velocity ω/k is superluminal and the EM wave well verifies the assumption of adiabaticity. For $\omega < \omega_{pe}$, the EM wave is evanescent.

For $\vec{k}\times\vec{E} = \vec{0}$, we obtain the plasmon mode $\omega = \omega_{pe}$, but with no energy transport. It corresponds to Langmuir's oscillations.

3.3 Bohm-Gross and ionic acoustic modes

The plasma is assumed to be isotropic and **unmagnetized** ($B_0 = 0$). We neglect the ionic response ($\vec{J}_p \simeq \vec{0}$) and the ionic temperature ($T_p \simeq 0$). With these assumptions, the Eq. (22) leads to

$$\frac{\partial \vec{V}_e}{\partial t} + \left(\vec{V}_e\cdot\vec{\nabla}\right)\,\vec{V}_e = -\frac{\vec{\nabla}P_e}{n_e\,m_e} - \frac{e}{m_e}\,\left(\vec{E}+\vec{V}_e\times\vec{B}\right)\,, \tag{65}$$

and we suppose the isentropic state equation (37). According to the linearization process and the continuity equation (giving the Eq. (42)), we obtain first

$$p = \frac{\gamma}{\omega}\, P_e\, \vec{k} \cdot \vec{V}_e\,, \tag{66}$$

with $P_e = n_e\, k_B\, T_e$, where k_B is the Boltzmann constant. Thus, the Eq. (65) can be written

$$\vec{J}_e = -i\, \frac{\omega_{pe}^2}{\omega}\, \epsilon_0\, \vec{E} + \frac{\gamma\, k_B\, T_e}{m_e\, \omega^2}\, \left(\vec{k} \cdot \vec{J}_e\right)\, \vec{k}. \tag{67}$$

For $\vec{k} \times \vec{E} = \vec{0}$, we obtain the Bohm-Gross mode: indeed, because of the previous equation, $\vec{k} \times \vec{J}_e = \vec{0}$, and we have

$$\vec{J}_e = -i\, \frac{\omega_{pe}^2}{\omega}\, \frac{\epsilon_0\, \vec{E}}{1 - \frac{\gamma\, k_B\, T_e\, k^2}{m_e\, \omega^2}}\,. \tag{68}$$

The dispersion relation of the *Bohm-Gross mode* is thus

$$\omega^2 = \omega_{pe}^2 + \frac{\gamma\, k_B\, T_e}{m_e}\, k^2\,. \tag{69}$$

For $\omega < \omega_{pe}$, there is no pure electronic mode. We have to take into account the ionic dynamics. We first assume low frequencies such that

$$\frac{k_B\, T_p}{m_p} < \frac{\omega^2}{k^2} < \frac{k_B\, T_e}{m_e}\,. \tag{70}$$

This is at the limit of the first adiabatic assumption (see Sec. 3.1). We neglect the electronic inertia. The Eq. (68) gives

$$\vec{J}_e \simeq -i\, \omega_{pe}^2\, \frac{m_e\, \omega\, \epsilon_0\, \vec{E}}{\gamma\, k_B\, T_e\, k^2}\,. \tag{71}$$

Concerning the ionic current, the Eq. (23) (with $T_p \simeq 0$) and the linearization process lead to

$$\vec{J}_p = -i\, \frac{\omega_{pp}^2}{\omega}\, \epsilon_0\, \vec{E}\,. \tag{72}$$

For low frequencies $\omega \ll \omega_{pp}$, one can assume the quasi-neutrality of current: $\vec{J}_e + \vec{J}_p \simeq \vec{0}$. Indeed, because $\vec{\nabla} \times \vec{B} = 0$, we have $\vec{J}_e + \vec{J}_p + i\, \omega\, \epsilon_0\, \vec{E} = \vec{0}$ and

$$\frac{|\omega\, \epsilon_0\, \vec{E}|}{|\vec{J}_e + \vec{J}_p|} \simeq \frac{\omega^2}{\omega_{pp}^2}\, \left[1 - \frac{m_e\, \omega_{pe}^2}{\gamma\, k_B\, T_e\, k^2}\, \frac{\omega^2}{\omega_{pp}^2} + o\left(\frac{\omega^2}{\omega_{pp}^2}\right)\right]\,. \tag{73}$$

We then deduce the following dispersion relation for the *phonon mode*;

$$\omega = C_s\, k\,, \tag{74}$$

with $C_s = \sqrt{\gamma\, k_B\, T_e/m_p}$.

For higher frequencies such that $\omega_{pp} > \omega > k\,\sqrt{k_B\, T_e/m_e}$, we cannot assume the quasi-neutrality. The Eqs. (68) and (72) yield

$$\frac{\omega_{pp}^2}{\omega^2} + \frac{\omega_{pp}^2}{(m_e/m_p)\omega^2 - k^2\, C_s^2} = 1\,, \tag{75}$$

with $\omega_{pp}/\omega_{pe} = \sqrt{m_e/m_p}$. By neglecting the term $(m_e/m_p)\omega^2$, we obtain the dispersion relation of the *ionic acoustic mode*;

$$\omega \simeq \omega_{pp}\, \frac{C_s\, k}{\sqrt{\omega_{pp}^2 + C_s^2\, k^2}}\,. \tag{76}$$

In this mode, we cannot have $\omega_{pp}^2 \ll C_s^2\, k^2$ which leads to $\omega \simeq \omega_{pp}$, because the condition $\omega/k \gg \sqrt{k_B\, T_e/m_p}$ would not be satisfied and we would have to take into account Landau's damping.

3.4 Magnetized electronic and ionic modes

The plasma is assumed to be **magnetized**. We will define the basis $\{\vec{e}_\parallel, \vec{e}_\perp, \vec{e}_\times\}$ as described in Sec. 2.3.

3.4.1 Electronic modes

We first neglect the ionic response ($\vec{J}_p \simeq \vec{0}$) and the temperature ($T_e = T_p \simeq 0$). With these assumptions, the Eq. (22) leads to

$$\frac{\partial \vec{V}_e}{\partial t} + \left(\vec{V}_e \cdot \vec{\nabla}\right) \vec{V}_e = -\frac{e}{m_e}\left(\vec{E} + \vec{V}_e \times \vec{B}\right)\,, \tag{77}$$

and the linearization process yields

$$\vec{J}_e = -i\,\frac{\omega_{pe}^2}{\omega}\,\epsilon_0\,\vec{E} + i\,\frac{\omega_{ce}}{\omega}\,\vec{J}_e \times \vec{e}_\parallel\,. \tag{78}$$

Because

$$\begin{aligned} \vec{J}_e \cdot \vec{e}_\parallel &= -i\,\frac{\omega_{pe}^2}{\omega}\,\epsilon_0\,\vec{E}\cdot\vec{e}_\parallel\,, && (79)\\ \vec{J}_e \times \vec{e}_\parallel &= -i\,\frac{\omega_{pe}^2}{\omega}\,\epsilon_0\,\vec{E}\times\vec{e}_\parallel - i\,\frac{\omega_{ce}}{\omega}\,\vec{J}_e + i\,\frac{\omega_{ce}}{\omega}\,\left(\vec{J}_e\cdot\vec{e}_\parallel\right)\,\vec{e}_\parallel\,, && (80)\end{aligned}$$

we deduce

$$\vec{J}_e = -i\,\frac{\omega_{pe}^2\,\omega}{\omega^2-\omega_{ce}^2}\,\left[\epsilon_0\,\vec{E} + i\,\frac{\omega_{ce}}{\omega}\,\epsilon_0\,\vec{E}\times\vec{e}_\parallel - \frac{\omega_{ce}^2}{\omega^2}\,\epsilon_0\,\left(\vec{E}\cdot\vec{e}_\parallel\right)\,\vec{e}_\parallel\right]\,. \qquad (81)$$

The Eqs. (3) and (4) gives

$$\vec{k}\times\left(\vec{k}\times\vec{E}\right) + \frac{\omega^2}{c^2}\,\left[\epsilon_\perp(\omega)\,\vec{E} + \frac{\omega_{ce}}{\omega}\,\epsilon_\times(\omega)\,\left(\vec{E}\cdot\vec{e}_\parallel\right)\,\vec{e}_\parallel - i\,\epsilon_\times(\omega)\,\vec{E}\times\vec{e}_\parallel\right] = \vec{0} \qquad (82)$$

with the coefficients,

$$\begin{aligned} \epsilon_\perp(\omega) &= 1 - \frac{\omega_{pe}^2}{\omega^2-\omega_{ce}^2}\,, && (83)\\ \epsilon_\times(\omega) &= \frac{\omega_{ce}}{\omega}\,\frac{\omega_{pe}^2}{\omega^2-\omega_{ce}^2}\,. && (84)\end{aligned}$$

By defining $\vec{E} = \vec{E}_\parallel + \vec{E}_\perp$ with $\vec{E}_\parallel = E_\parallel\,\vec{e}_\parallel$ and $\vec{E}_\perp\cdot\vec{e}_\parallel = 0$, we obtain

$$\vec{k}\times\left(\vec{k}\times\vec{E}\right) + \frac{\omega^2}{c^2}\,\left[\epsilon_\parallel(\omega)\,\vec{E}_\parallel + \epsilon_\perp(\omega)\,\vec{E}_\perp - i\,\epsilon_\times(\omega)\,\vec{E}_\perp\times\,\vec{e}_\parallel\right] = \vec{0}\,, \qquad (85)$$

with the last coefficient,

$$\epsilon_\parallel(\omega) = 1 - \frac{\omega_{pe}^2}{\omega^2}\,. \qquad (86)$$

(1) For a parallel propagation i.e $\vec{k} = k_\parallel\,\vec{e}_\parallel$, the Eq. (85) is equivalent to

$$\begin{pmatrix} \frac{\omega^2}{c^2}\,\epsilon_\parallel & 0 & 0 \\ 0 & \frac{\omega^2}{c^2}\,\epsilon_\perp - k_\parallel^2 & -i\,\frac{\omega^2}{c^2}\,\epsilon_\times \\ 0 & i\,\frac{\omega^2}{c^2}\,\epsilon_\times & \frac{\omega^2}{c^2}\,\epsilon_\perp - k_\parallel^2 \end{pmatrix} \begin{pmatrix} E_\parallel \\ E_{\perp y} \\ E_{\perp z} \end{pmatrix} = 0\,, \qquad (87)$$

in the basis $\{\vec{e}_\parallel, \vec{e}_\perp, \vec{e}_\times\}$. For $\vec{E} = \vec{E}_\parallel$, we recognize the plasmon mode (see Sec. 3.2) with $\omega = \omega_{pe}$. For $\vec{E} \neq \vec{E}_\parallel$, the other proper modes are obtained for det $= 0$ i.e $k_\parallel^2\, c^2 = \omega^2\,(\epsilon_\perp \pm \epsilon_\times)$. The two dispersion relations correspond to the following modes:

(1-a) The *left mode* (L-mode):

$$k_\parallel^2 = \frac{\omega^2}{c^2}\left[1 - \frac{\omega_{pe}^2}{\omega\,(\omega + \omega_{ce})}\right], \tag{88}$$

with the cut-off frequency ($k_\parallel = 0$) such that

$$\omega_L = \sqrt{\omega_{pe}^2 + \frac{\omega_{ce}^2}{4}} - \frac{\omega_{ce}}{2}\,. \tag{89}$$

(1-b) The *right mode* (R-mode):

$$k_\parallel^2 = \frac{\omega^2}{c^2}\left[1 - \frac{\omega_{pe}^2}{\omega\,(\omega - \omega_{ce})}\right], \tag{90}$$

with the cut-off frequency ($k_\parallel = 0$) such that

$$\omega_R = \sqrt{\omega_{pe}^2 + \frac{\omega_{ce}^2}{4}} + \frac{\omega_{ce}}{2}\,, \tag{91}$$

and the resonance frequency ($k_\parallel \to +\infty$) for $\omega \to \omega_{ce}$.
We can plot two different Brillouin diagrams (ω-k plane) for the R-mode: for a strong magnetized plasma, $\omega_{ce} > \omega_{pe}$, and for a dense plasma, $\omega_{ce} < \omega_{pe}$.

The different phase velocity between the R-mode and the L-mode is at the origin of the Faraday effect, for a rectilinear polarized wave, which is the rotation of the rectilinear polarization plane.

For $\omega \ll \omega_{ce}$, the R-mode is named *whistler mode* and the dispersion relation is simply:

$$\omega = k_\parallel^2\, c^2\, \frac{\omega_{ce}}{\omega_{pe}^2}\,. \tag{92}$$

The group velocity is thus $v_g = \partial\omega/\partial k_\parallel \propto \sqrt{\omega}$.

(2) For a perpendicular propagation i.e, for example, $\vec{k} = k_\perp \vec{e}_\perp$, the Eq. (85) is equivalent to

$$\begin{pmatrix} \frac{\omega^2}{c^2}\epsilon_\parallel - k_\perp^2 & 0 & 0 \\ 0 & \frac{\omega^2}{c^2}\epsilon_\perp & -i\,\frac{\omega^2}{c^2}\epsilon_\times \\ 0 & i\,\frac{\omega^2}{c^2}\epsilon_\times & \frac{\omega^2}{c^2}\epsilon_\perp - k_\perp^2 \end{pmatrix} \begin{pmatrix} E_\parallel \\ E_{\perp y} \\ E_{\perp z} \end{pmatrix} = 0\,, \tag{93}$$

in the basis $\{\vec{e}_\parallel, \vec{e}_\perp, \vec{e}_\times\}$. For $\vec{E}_\perp = \vec{0}$, we obtain the *ordinary mode* with the dispersion relation:

$$\omega^2 = \omega_{pe}^2 + k_\perp^2\, c^2\,. \tag{94}$$

This is similar to the electromagnetic mode previously obtained in Sec. 3.2. For $\vec{E}_\perp \neq \vec{0}$, we deduce from det $= 0$, the *extraordinary mode* with the following dispersion relation:

$$k_\perp^2\, c^2 = \omega^2\,(\epsilon_\perp - \epsilon_\times^2/\epsilon_\perp)\,, \tag{95}$$

which can be written

$$k_\perp^2\, c^2 = \frac{(\omega^2 - \omega_R^2)\,(\omega^2 - \omega_L^2)}{\omega^2 - \omega_{UH}^2}\,, \tag{96}$$

where the frequency of the high hybrid resonance, corresponding to the Langmuir electronic oscillation, is such that

$$\omega_{UH}^2 = \frac{1}{2}\left[\omega_{ce}^2 + \omega_{pe}^2 + \sqrt{\omega_{ce}^2 + \omega_{pe}^2 - 4\,\omega_{ce}^2\,(\omega_{cp}^2 + \omega_{pp}^2)}\right]\,. \tag{97}$$

For $\omega_{pe} \gg \omega_{pp}$ and, because $\omega_{ce} \gg \omega_{cp}$, $\omega_{UH} \simeq \sqrt{\omega_{ce}^2 + \omega_{pe}^2}$.

(3) For an oblique propagation, $\vec{k} = k_\perp \vec{e}_\perp + k_\parallel \vec{e}_\parallel$, with $k_\perp = k\,\sin\theta$ and $k_\parallel = k\,\cos\theta$. The Eq. (85) is equivalent to

$$\begin{pmatrix} \frac{\omega^2}{c^2}\epsilon_\parallel - k_\perp^2 & 0 & k_\perp\, k_\parallel \\ 0 & \frac{\omega^2}{c^2}\epsilon_\perp - k_\parallel^2 - k_\perp^2 & -i\,\frac{\omega^2}{c^2}\epsilon_\times \\ k_\perp\, k_\parallel & i\,\frac{\omega^2}{c^2}\epsilon_\times & \frac{\omega^2}{c^2}\epsilon_\perp - k_\parallel^2 \end{pmatrix} \begin{pmatrix} E_\parallel \\ E_{\perp y} \\ E_{\perp z} \end{pmatrix} = 0\,, \tag{98}$$

in the basis $\{\vec{e}_\parallel, \vec{e}_\perp, \vec{e}_\times\}$. By defining,

$$a_1(\theta) = \epsilon_\perp\,\sin^2\theta + \epsilon_\parallel\,\cos^2\theta\,, \tag{99}$$

$$a_2(\theta) = \epsilon_\perp\,\epsilon_\parallel\,(1 + \cos^2\theta) + (\epsilon_\perp^2 - \epsilon_\times^2)\,\sin^2\theta\,, \tag{100}$$

$$a_3 = \epsilon_\parallel\,(\epsilon_\perp^2 - \epsilon_\times^2)\,, \tag{101}$$

we obtain the proper modes (det = 0) with the equation

$$a_1 \left(\frac{k\,c}{\omega}\right)^4 - a_2 \left(\frac{k\,c}{\omega}\right)^2 + a_3 = 0\,. \tag{102}$$

Thus, we have the dispersion relation

$$k = \frac{\omega}{c} \sqrt{\frac{a_2 \pm \sqrt{a_2^2 - 4a_1\,a_3}}{2a_1}}\,. \tag{103}$$

There are resonances ($k \to +\infty$) for $a_1 \to 0$ and cut-off ($k \to 0$) for $a_3 \to 0$.

3.4.2 Ionic modes

We also consider a cold plasma by neglecting the temperature ($T_e = T_p \simeq 0$), but we introduce the ionic current ($\vec{J}_p \neq \vec{0}$). With these assumptions, the Eqs. (22) and (23), and the linearization process, yield

$$\vec{J}_e = -i\,\frac{\omega_{pe}^2\,\omega}{\omega^2 - \omega_{ce}^2} \left[\epsilon_0\,\vec{E} + i\,\frac{\omega_{ce}}{\omega}\,\epsilon_0\,\vec{E} \times \vec{e}_\parallel - \frac{\omega_{ce}^2}{\omega^2}\,\epsilon_0 \left(\vec{E} \cdot \vec{e}_\parallel\right) \vec{e}_\parallel\right], \tag{104}$$

$$\vec{J}_p = -i\,\frac{\omega_{pp}^2\,\omega}{\omega^2 - \omega_{cp}^2} \left[\epsilon_0\,\vec{E} - i\,\frac{\omega_{cp}}{\omega}\,\epsilon_0\,\vec{E} \times \vec{e}_\parallel - \frac{\omega_{cp}^2}{\omega^2}\,\epsilon_0 \left(\vec{E} \cdot \vec{e}_\parallel\right) \vec{e}_\parallel\right], \tag{105}$$

and we obtain the same relation than in the previous section,

$$\vec{k} \times \left(\vec{k} \times \vec{E}\right) + \frac{\omega^2}{c^2} \left[\epsilon_\parallel(\omega)\,\vec{E}_\parallel + \epsilon_\perp(\omega)\,\vec{E}_\perp - i\,\epsilon_\times(\omega)\,\vec{E}_\perp \times \vec{e}_\parallel\right] = \vec{0}\,, \tag{106}$$

but with the following new definitions for coefficients

$$\epsilon_\perp(\omega) = 1 - \frac{\omega_{pe}^2}{\omega^2 - \omega_{ce}^2} - \frac{\omega_{pp}^2}{\omega^2 - \omega_{cp}^2}\,, \tag{107}$$

$$\epsilon_\times(\omega) = \frac{\omega_{ce}}{\omega}\,\frac{\omega_{pe}^2}{\omega^2 - \omega_{ce}^2} - \frac{\omega_{cp}}{\omega}\,\frac{\omega_{pp}^2}{\omega^2 - \omega_{cp}^2}\,, \tag{108}$$

$$\epsilon_\parallel(\omega) = 1 - \frac{\omega_{pe}^2}{\omega^2} - \frac{\omega_{pp}^2}{\omega^2}\,. \tag{109}$$

For high frequencies ($\omega \gg \omega_{cp}$), the situation is the same one than in the Sec. 3.4.1.

For very low frequencies ($\omega \to 0$), because $\omega_{ce} \gg \omega_{cp}$, we have $\epsilon_\parallel \to -\omega_{pe}^2/\omega$, $\epsilon_\perp \to 1 + c^2/V_A^2$ and, $\epsilon_\times \to (c^2/V_A^2)\,\omega/\omega_{cp}$.

(1) For a parallel propagation i.e $\vec{k} = k_\parallel\,\vec{e}_\parallel$, and for $\vec{E} \neq \vec{E}_\parallel$, the Eq. (106) leads to $k_\parallel^2\,c^2 = \omega^2\,(\epsilon_\perp \pm \epsilon_\times)$. For low frequencies ($\omega \ll \omega_{pp}$), and for $n_e \simeq n_p$ (i.e $\omega_{pp}^2/\omega_{cp} \simeq \omega_{pe}^2/\omega_{ce}$), we deduce

$$\omega^2 = k_\parallel^2\,V_A^2\,\left[1 \pm \frac{\omega}{\omega_{cp}} + o\left(\frac{\omega}{\omega_{cp}}\right)\right]\,. \tag{110}$$

We thus define again the following modes:

(1-a) The *left mode* (L-mode):

$$\omega \simeq k_\parallel\,V_A\,\sqrt{1 - \frac{\omega}{\omega_{cp}}}\,. \tag{111}$$

(1-b) The *right mode* (R-mode):

$$\omega \simeq k_\parallel\,V_A\,\sqrt{1 + \frac{\omega}{\omega_{cp}}}\,. \tag{112}$$

For $\omega \to 0$, these two modes blend into the Alfvén mode, as previously described in Sec. 2.3: $\omega \simeq k_\parallel\,V_A$.

(2) For a perpendicular propagation i.e, for example, $\vec{k} = k_\perp\,\vec{e}_\perp$, and for $\vec{E}_\perp \neq \vec{0}$, we deduce from the Eq. (106) the dispersion relation for the *extraordinary mode*:

$$k_\perp^2\,c^2 = \omega^2\,(\epsilon_\perp - \epsilon_\times^2/\epsilon_\perp)\,, \tag{113}$$

which leads to

$$k_\perp^2\,c^2 = \omega^2\,\frac{(\omega^2 - \omega_R^2)\,(\omega^2 - \omega_L^2)}{(\omega^2 - \omega_{LH}^2)(\omega^2 - \omega_{UH}^2)}\,, \tag{114}$$

where the frequency of the low hybrid resonance, corresponding to the Langmuir oscillation, is such that

$$\omega_{LH}^2 = \frac{1}{2}\,\left[\omega_{ce}^2 + \omega_{pe}^2 - \sqrt{\omega_{ce}^2 + \omega_{pe}^2 - 4\,\omega_{ce}^2\,(\omega_{cp}^2 + \omega_{pp}^2)}\right]\,. \tag{115}$$

For $\omega_{pe} \gg \omega_{pp}$ and, because $\omega_{ce} \gg \omega_{cp}$, we have

$$\omega_{LH} \simeq \omega_{ce}\,\sqrt{\frac{\omega_{cp}^2 + \omega_{pp}^2}{\omega_{ce}^2 + \omega_{pe}^2}} \simeq \sqrt{\omega_{ce}\,\omega_{cp}}. \tag{116}$$

For $\omega \longrightarrow 0$, we recognize again the Alfvén mode because $\omega \simeq k_\perp V_A/\sqrt{1+V_A^2/c^2} \simeq k_\perp V_A$ (with $V_A \ll c$).

(3) If we slightly modify the assumptions by taking into account the electronic compressibility effects ($T_e \neq 0$), and if we consider pulsations ω such that $\omega_{cp} < \omega < \omega_{ce}$, the modification of the Eq. (104) with the pressure term, and the Eq. (105) give

$$\vec{J}_{e\|} \simeq -i\,\frac{\omega_{pe}^2}{\omega}\,\epsilon_0\,\vec{E}_\| + \frac{\gamma\,k_B\,T_e\,k^2}{m_e\,\omega^2}\,\vec{J}_{e\|}\,, \tag{117}$$

$$\vec{J}_{p\|} \simeq -i\,\frac{\omega_{pp}^2\,\omega}{\omega^2-\omega_{cp}^2}\,\epsilon_0\,\vec{E}_\|\,, \tag{118}$$

with $\omega_{ce}^2/\omega^2 \gg 1$ and, by neglecting terms in ω_{cp}/ω. Because $\omega/k \ll \sqrt{\gamma\,k_B\,T_e/m_e}$, the Eq. (117) can be written

$$\vec{J}_{e\|} \simeq i\,\frac{\omega_{pe}^2\,\omega\,m_e}{\gamma\,k_B\,T_e\,k^2}\,\epsilon_0\,\vec{E}_\|\,. \tag{119}$$

The quasi-neutrality of current, $\vec{J}_{e\|} + \vec{J}_{p\|} = \vec{0}$, leads to dispersion relation of the *ionic cyclotron mode*;

$$\omega^2 = \omega_{cp}^2 + C_s^2\,k^2\,, \tag{120}$$

with $C_s = \sqrt{\gamma\,k_B\,T_e/m_p}$, and because $\omega_{pp}^2/\omega_{pe}^2 = m_e/m_p$.

4 Summary

Let us summarize the previous results concerning waves propagating in an isotropic $p - e^-$ plasma.

I - For an unmagnetized plasma i.e $B_0 = 0$

I-1 The ionic response ($\vec{J}_p$) and the temperature ($T_e = T_p \simeq 0$) are neglected:

- **The electromagnetic mode** ($\vec{k} \cdot \vec{E} = 0$):

$$\omega^2 = \omega_{pe}^2 + k^2\, c^2\,. \tag{121}$$

For $\omega > \omega_{pe}$, the phase velocity ω/k is superluminal.
For $\omega < \omega_{pe}$, the EM wave is evanescent ($k = k_r + i\, k_i$, with $k_i \neq 0$).

- **The plasmon mode or Langmuir's oscillations** ($\vec{k} \times \vec{E} = \vec{0}$): $\omega = \omega_{pe}$. No energy transport.

I-2 The ionic response ($\vec{J}_p$) and the ionic temperature ($T_p \simeq 0$) are first neglected:

- **The Bohm-Gross mode** ($\vec{k} \times \vec{E} = \vec{0}$):

$$\omega^2 = \omega_{pe}^2 + \frac{\gamma\, k_B\, T_e}{m_e}\, k^2\,. \tag{122}$$

For $\omega < \omega_{pe}$, there is no pure electronic mode. We have to take into account the ionic dynamics ($\vec{J}_p$).

- **The phonon mode** ($\omega \ll \omega_{pp}$ and $\vec{J}_e + \vec{J}_p \simeq \vec{0}$):

$$\omega = C_s\, k\,, \tag{123}$$

with $C_s = \sqrt{\gamma\, k_B\, T_e/m_p}$.

- **The ionic acoustic mode** ($k\, \sqrt{k_B\, T_e/m_e} < \omega < \omega_{pp}$):

$$\omega \simeq \omega_{pp}\, \frac{C_s\, k}{\sqrt{\omega_{pp}^2 + C_s^2\, k^2}}\,. \tag{124}$$

II - For a magnetized plasma i.e $B_0 \neq 0$

II-1 **The electronic modes** ($\vec{J_p} \simeq \vec{0}$ and $T_e = T_p \simeq 0$):

- The **L-mode** ($\vec{k} \parallel \vec{B}$):

$$k_{\parallel}^2 = \frac{\omega^2}{c^2} \left[1 - \frac{\omega_{pe}^2}{\omega\,(\omega + \omega_{ce})}\right] , \tag{125}$$

with the cut-off frequency ($k_{\parallel} = 0$) such that

$$\omega_L = \sqrt{\omega_{pe}^2 + \frac{\omega_{ce}^2}{4}} - \frac{\omega_{ce}}{2} . \tag{126}$$

- The **R-mode** ($\vec{k} \parallel \vec{B}$):

$$k_{\parallel}^2 = \frac{\omega^2}{c^2} \left[1 - \frac{\omega_{pe}^2}{\omega\,(\omega - \omega_{ce})}\right] , \tag{127}$$

with the cut-off frequency ($k_{\parallel} = 0$) such that

$$\omega_R = \sqrt{\omega_{pe}^2 + \frac{\omega_{ce}^2}{4}} + \frac{\omega_{ce}}{2} , \tag{128}$$

and the resonance frequency ($k_{\parallel} \to +\infty$) for $\omega \to \omega_{ce}$.
For $\omega \ll \omega_{ce}$, the R-mode is named **whistler mode** and the dispersion relation is simply:

$$\omega = k_{\parallel}^2\, c^2\, \frac{\omega_{ce}}{\omega_{pe}^2} . \tag{129}$$

- The **ordinary mode** equivalent to the electromagnetic mode ($\vec{E}_{\perp} = \vec{0}$ and $\vec{k} \perp \vec{B}$):

$$\omega^2 = \omega_{pe}^2 + k_{\perp}^2\, c^2 . \tag{130}$$

- The **extraordinary mode** ($\vec{E}_{\perp} \neq \vec{0}$ and $\vec{k} \perp \vec{B}$):

$$k_{\perp}^2\, c^2 = \frac{(\omega^2 - \omega_R^2)\,(\omega^2 - \omega_L^2)}{\omega^2 - \omega_{UH}^2} , \tag{131}$$

with the frequency of the high hybrid resonance,

$$\omega_{UH}^2 = \frac{1}{2}\left[\omega_{ce}^2 + \omega_{pe}^2 + \sqrt{\omega_{ce}^2 + \omega_{pe}^2 - 4\,\omega_{ce}^2\,(\omega_{cp}^2 + \omega_{pp}^2)}\right] . \tag{132}$$

For $\omega_{pe} \gg \omega_{pp}$, $\omega_{UH} \simeq \sqrt{\omega_{ce}^2 + \omega_{pe}^2}$.

II-2 **The ionic modes** ($\vec{J_p} \neq \vec{0}$ and $T_e = T_p \simeq 0$):

- The **L-mode** ($\vec{k} \parallel \vec{B}$):

$$\omega \simeq k_{\parallel} V_A \sqrt{1 - \frac{\omega}{\omega_{cp}}} \,. \tag{133}$$

- The **R-mode** ($\vec{k} \parallel \vec{B}$):

$$\omega \simeq k_{\parallel} V_A \sqrt{1 + \frac{\omega}{\omega_{cp}}} \,. \tag{134}$$

For $\omega \rightarrow 0$, these two modes blend into the Alfvén mode: $\omega \simeq k_{\parallel} V_A$.

- The **extraordinary mode** ($\vec{E}_{\perp} \neq \vec{0}$ and $\vec{k} \perp \vec{B}$):

$$k_{\perp}^2 c^2 = \omega^2 \frac{(\omega^2 - \omega_R^2)(\omega^2 - \omega_L^2)}{(\omega^2 - \omega_{LH}^2)(\omega^2 - \omega_{UH}^2)} \,, \tag{135}$$

with the frequency of the low hybrid resonance,

$$\omega_{LH}^2 = \frac{1}{2} \left[\omega_{ce}^2 + \omega_{pe}^2 - \sqrt{\omega_{ce}^2 + \omega_{pe}^2 - 4\,\omega_{ce}^2\,(\omega_{cp}^2 + \omega_{pp}^2)} \right] \,. \tag{136}$$

For $\omega_{pe} \gg \omega_{pp}$ and, because $\omega_{ce} \gg \omega_{cp}$, we have

$$\omega_{LH} \simeq \omega_{ce} \sqrt{\frac{\omega_{cp}^2 + \omega_{pp}^2}{\omega_{ce}^2 + \omega_{pe}^2}} \simeq \sqrt{\omega_{ce}\,\omega_{cp}}. \tag{137}$$

For $\omega \rightarrow 0$ (or $\omega \ll \omega_{cp}$), we obtain the Alfvén mode because $\omega \simeq k_{\perp} V_A / \sqrt{1 + V_A^2/c^2} \simeq k_{\perp} V_A$ (with $V_A \ll c$).

- The **slow (-) and fast (+) magnetosonic modes** ($\omega \ll \omega_{cp}$):

$$\omega_{f,s}^2 = \frac{k^2}{2} \left[(C_s^2 + V_A^2) \pm \sqrt{(C_s^2 + V_A^2)^2 - 4\,C_s^2 V_A^2 \left(\frac{k_{\parallel}}{k}\right)^2} \right] \,. \tag{138}$$